Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3204 Homological Algebra Fall 2018

Exercise set 6

1 Given a right R-module M, the *trace ideal*, $\operatorname{tr}(M)$, is the right ideal generated by all $f(M)$ for $f \in \operatorname{Hom}_R(M, R)$:

$$\operatorname{tr}(M) := \langle f(x) \mid f \in \operatorname{Hom}_R(M, R),\, x \in M \rangle.$$

Show that the following are equivalent for an R-module G:

1. G is a generator in $\mathbf{Mod}\, R$;
2. $\operatorname{tr}(G) = R$;
3. R is a direct summand of a finite direct sum $\oplus_i G$;
4. R is a direct summand of a direct sum $\oplus_i G$;
5. Every R-module M is an epimorphic image of some direct sum $\oplus_i G$.

2 Let R be a ring. An idempotent $e \in R$ is called *full* if $ReR = R$.

a) Show that if e is a full idempotent, then eR is a progenerator in $\mathbf{Mod}\, R$ and R is Morita equivalent to eRe.

Let $M_n(R) \cong \operatorname{End}_R(R^n)$ be the ring of $n \times n$ matrices with entries in R.

b) Let P be a finitely generated projective R-module. We have seen that $P \oplus Q \cong R^n$ for some module Q and $n \geq 1$. In other words, $P \cong eR^n$, where $e = (a_{ij}) \in M_n(R)$ is the idempotent given by the projection $e : R^n \to R^n$ of R^n onto P, defined as $e_{|_P} = \operatorname{id}_P$ and $e_{|_Q} = 0$. Show that

$$\operatorname{tr}(P) = \sum_{i,j=1}^{n} R a_{ij} R \quad \text{and} \quad M_n(R) e M_n(R) = M_n(\operatorname{tr}(P)).$$

Conclude that P is a progenerator if and only if e is a full idempotent in $M_n(R)$.
Hints: Let $f_j \in \operatorname{Hom}_R(P, R)$ be the map that sends a vector in P to its jth component. Then any $f \in \operatorname{Hom}_R(P, R)$ is of the form $f = \sum_j \mu_j f_j$ for some $\mu_j \in R$. Show that for any $x \in P$,

$$f(x) = \sum_{i,j=1}^{n} \mu_i f_i(a_j) f_j(x),$$

where a_j is the jth column of e.
For the ideal $M_n(R) e M_n(R)$, consider elements of the form

$$r E_{ij} e E_{kl} r' = r a_{jk} r' E_{il},$$

where the matrices E_{ij} have 1 in the (i, j)-entry and 0 elsewhere.

c) Show that two rings R and S are Morita equivalent if and only if $S \cong eM_n(R)e$ for some $n \geq 1$ and a full idempotent $e \in M_n(R)$.

3 The goal of this exercise is to fill in the blanks in the proof of the Eilenberg-Watts theorem. Let $F : \mathbf{Mod}\, R \to \mathbf{Mod}\, S$ be an additive functor.

a) Show that the S-module FR can be given a left R-module structure that makes it into a $(R$-$S)$-bimodule.

b) Given an R-module M, let

$$\begin{aligned} \Psi'_M : M \times FR &\longrightarrow FM \\ (m, \bar{r}) &\longmapsto (F\phi_m)(\bar{r}), \end{aligned}$$

where

$$\begin{aligned} \phi_m : R &\longrightarrow M \\ s &\longmapsto ms. \end{aligned}$$

Show that Ψ'_M induces a unique R-module homomorphism $\Psi_M : M \otimes_R FR \to FM$ and that $\Psi = (\Psi_M)_{M \in \mathbf{Mod}\, R}$ is a natural transformation $- \otimes_R FR \Longrightarrow F$.