Norwegian University of Science and Technology
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MA3204 Homological Algebra Fall 2018

Exercise set 5

1 **a)** Let R be a commutative ring and I, J two ideals in R. Show that

$$R/I \otimes_R R/J \cong R/(I+J).$$

b) Deduce that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

2 Let R be a commutative ring. Recall that an R-*algebra* A is a ring with the structure of an R-module such that the scalar multiplication satisfies

$$r \cdot (xy) = (r \cdot x)y = x(r \cdot y)$$

for all $r \in R$ and $x, y \in A$.

a) If A and B are two R-algebras, show that $A \otimes_R B$ is a R-algebra with multiplication $(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'$.

b) Show that the polynomial ring $R[x, y] \cong R[x] \otimes_R R[y]$ as R-algebras.

3 Let R be a domain. An R-module M is called *torsion-free* if for all $0 \neq m \in M$, $mr = 0$ for some $r \in R$ implies that $r = 0$.

a) Show that if M is a flat R-module, then M is torsion-free.

b) Show that the converse is true if R is a PID. **Hint**: First show that a module M is flat if and only all of its finitely generated submodules are flat. Then use the structure theorem for finitely generated modules over a PID.

4 A short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called *split exact* if one of the following conditions is satisfied:

- f is a split monomorphism;
- g is a split epimorphism;
- There exist morphisms

$$A \underset{k}{\overset{f}{\rightleftarrows}} B \cong A \oplus C \underset{g}{\overset{h}{\leftrightarrows}} C$$

  satisfying the biproduct identities.

Show that these conditions are indeed equivalent.

Remark: If $B \cong A \oplus C$ as objects, then the short exact sequence is not necessarily split. It needs to be an isomorphism at the level of biproducts, i.-e. it must also come with maps satisfying the biproduct properties. This is similiar to the fact that objects isomorphic to kernels and limits are not necessarily kernels and limits themselves; they must come with morphisms and respect the universal properties. For example, the following short exact sequence is not split:

$$0 \to \mathbb{Z} \oplus 0 \xrightarrow{(\cdot 2, \cdot 0)} \mathbb{Z} \oplus \coprod_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\pi, \mathrm{id})} \mathbb{Z}/2\mathbb{Z} \oplus \coprod_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to 0$$

even if the middle term is isomorphic as an abelian group to the direct sum of the two outer terms.

5 Let $(F, G) : \mathscr{A} \to \mathscr{B}$ be an adjoint pair between abelian categories. Show that

a) F is right exact and G is left exact;

b) If G is right exact, then F preserves projective objects;

c) If F is left exact, then G preserves injective objects.

Remark: This result can be used to show that if M and N are projective R-modules for some commutative ring R, then so is $M \otimes_R N$. Indeed $- \otimes_R N$ is left adjoint to $\mathrm{Hom}_R(N, -)$, which is exact by definition of projectivity.