Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3204
Homological Algebra
Fall 2018

Exercise set 4

1 Show that a functor $F : \mathscr{A} \to \mathscr{B}$ between additive categories is additive if and only if it preserves biproducts.

2 In an abelian category $\mathscr{A}$ consider a morphism $f : X \to Y$ and an epimorphism $g : W \twoheadrightarrow X$.

a) Show that $\operatorname{Im}(f \circ g) \cong \operatorname{Im} f$.

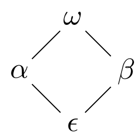
b) Show that $\operatorname{Cok}(f \circ g) \cong \operatorname{Cok} f$.

c) Assume $f = m \circ e$, with $e : X \to I$ an epimorphism and $m : I \to Y$ a monomorphism. Show that $I \cong \operatorname{Im} f$.

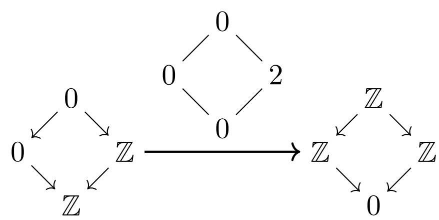
3 **a)** Show that for any small category $\mathscr{X}$, and any additive category $\mathscr{A}$, the category $\operatorname{presh}_{\mathscr{A}} \mathscr{X}$ is additive.

b) Show that for any small category $\mathscr{X}$, and any abelian category $\mathscr{A}$, the category $\operatorname{presh}_{\mathscr{A}} \mathscr{X}$ is abelian.

Remark: To build intuition, you may want to consider the following example. Take $\mathscr{A} = \mathbf{Ab}$ and let X be the poset



Consider the morphism (i.-e. natural transformation) in $\operatorname{presh}_{\mathbf{Ab}} X$:



where "2" denotes multiplication by 2. What are the kernel, image and cokernel of the morphism?

4 (Third isomorphism theorem) In an abelian category, let $f : A \to B$ and $g : B \to C$ be monomorphisms. Show that there is a short exact sequence

$$0 \longrightarrow \operatorname{Cok} f \longrightarrow \operatorname{Cok} g \circ f \longrightarrow \operatorname{Cok} g \longrightarrow 0.$$

Remark: In **Ab**, we have $\operatorname{Cok} f = B/A$, $\operatorname{Cok} g \circ f = C/A$ and $\operatorname{Cok} g = C/B$. The short exact sequence implies that $\operatorname{Cok} g$ is the cokernel of the monomorphism $\operatorname{Cok} f \longrightarrow \operatorname{Cok} g \circ f$, that is,

$$\frac{C/A}{B/A} \cong C/B.$$

5 (3×3 lemma) Consider the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & & B & & C & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Show that A, B and C also form a short exact sequence fitting into the above commutative diagram.