Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3204 Homological Algebra Fall 2018

Exercise set 3

1 **a)** Let f : **Ab** $\to$ **Gp** be the forgetful functor.

1. Show that the functor

$$\begin{aligned} F : \mathbf{Gp} &\longrightarrow \mathbf{Ab} \\ G &\longmapsto G^{\mathrm{ab}}, \end{aligned}$$

where $G^{\mathrm{ab}} := G/[G, G]$, is a left adjoint to f. Determine the unit and counit of the adjunction. Deduce the universal property of the abelianization of groups.

2. Show that f does not have a right adjoint.

b) Let f : **Top** $\to$ **Set** be the forgetful functor. Construct a right adjoint to f. Give the unit and counit of the adjunction.

c) Let **Rng** denote the category of associative non-unital rings.

1. For $R \in \mathbf{Rng}$, let R_+ have underlying set $R \times \mathbb{Z}$, and define addition and multiplication as $(r, n) + (r', n') = (r + r', n + n')$ and $(r, n) \cdot (r', n') = (rr' + rn' + r'n, nn')$. Show that R_+ is a unital ring.

2. Show that $j : R \to R_+ : r \to (r, 0)$ is a ring homomorphism satisfying the following universal property: For each unital ring S and ring homomorphism $f : R \to S$ there exists a unique unital ring homomorphism $\bar{f} : R_+ \to S$ such that $\bar{f}j = f$.

3. Show that $(-)_+ : \mathbf{Rng} \to \mathbf{Ring}$ is a functor, which is furthermore left adjoint to the forgetful functor f : **Ring** $\to$ **Rng**. Determine the unit and the counit of the adjunction.

2 In the category **Ab**.

a) Show that the pullback of

$$\begin{array}{ccc} & & L \\ & & \downarrow{\scriptstyle \alpha} \\ M & \xrightarrow{\beta} & N \end{array}$$

is given by $L \prod_N M = \{(l, m) \in L \oplus M \mid \alpha(l) = \beta(m)\}$.

b) Show that the pushout of

$$\begin{array}{ccc} L & \xrightarrow{\beta} & M \\ \downarrow{\scriptstyle\alpha} & & \\ N & & \end{array}$$

is given by $M \coprod_L N = (M \oplus N)/\{(\beta(l), -\alpha(l)) \mid l \in L\}$.

3 **a)** (Universal property of adjunction) Show that a functor $F : \mathscr{C} \to \mathscr{D}$ is a left adjoint functor if and only if for every object $D \in \mathbf{Ob}\,\mathscr{D}$, there exists an object $GD \in \mathbf{Ob}\,\mathscr{C}$ and a morphism $\epsilon_D : F(GD) \to D$, such that for every object $C \in \mathbf{Ob}\,\mathscr{C}$ and every morphism $f : F(C) \to D$, there exists a unique morphism $g : C \to GD$ with $\epsilon_D \circ Fg = f$.

b) Show that a functor $G : \mathscr{D} \to \mathscr{C}$ has a left adjoint if and only if $\mathrm{Hom}_{\mathscr{C}}(C, G-) : \mathscr{D} \to \mathbf{Set}$ is representable for all $C \in \mathbf{Ob}\,\mathscr{C}$.

4 Let $\mathscr{C}$ be a category whose objects are sets such that the forgetful functor $\mathrm{f} : \mathscr{C} \to \mathbf{Set}$ has a left adjoint. Show that a morphism in $\mathscr{C}$ is a monomorphism if and only if it is injective on the underlying sets.

Remark: In particular, **Set**, **Top**, **Gp**, **Ab**, **Ring** and **Mod**-R satisfy this property. The dual statement is also true: if the forgetful functor has a right adjoint then a morphism in $\mathscr{C}$ is an epimorphism if and only if it is surjective on the underlying sets. For **Gp**, **Ab**, **Ring** and **Mod**-R, the forgetful functor does not have a right adjoint. Still, in **Gp**, **Ab** and **Mod**-R, epimorphisms are exactly the surjective homomorphisms.