Norwegian University of Science and Technology
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MA3204 Homological Algebra Fall 2018

Exercise set 2

1 Let $\mathcal{C}$ be a category, $C \in \mathbf{Ob}\,\mathcal{C}$ and F a covariant functor $\mathcal{C} \to \mathbf{Set}$.

a) For $x \in FC$, define a family of maps $\xi^x = (\xi^x_D)_{D \in \mathbf{Ob}\,\mathcal{C}}$ as follows:

$$\xi^x_D : \mathrm{Hom}_{\mathcal{C}}(C, D) \to FD : f \mapsto (Ff)(x).$$

Show that ξ^x is a natural transformation $\mathrm{Hom}_{\mathcal{C}}(C, -) \Longrightarrow F$.

b) (Yoneda Lemma) Show that there is a bijection between the sets

$$\begin{aligned} \{\text{natural transformations } \mathrm{Hom}_{\mathcal{C}}(C, -) \Longrightarrow F\} &\longleftrightarrow FC \\ \eta &\longmapsto \eta_C(\mathrm{id}_C) \\ \xi^x &\longleftarrow\!\shortmid x. \end{aligned}$$

2 A covariant (resp. contravariant) functor $F : \mathcal{C} \to \mathbf{Set}$ is called *representable* if there exists $C \in \mathbf{Ob}\,C$ such that

$$F \cong \mathrm{Hom}_{\mathcal{C}}(C, -) \quad (\text{resp. } F \cong \mathrm{Hom}_{\mathcal{C}}(-, C)).$$

We then say that C *represents* the functor F. Note that a contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ is often called a *presheaf*.

a) Using Yoneda Lemma, show that a covariant functor is representable if and only if there exists $C \in \mathbf{Ob}\,\mathcal{C}$ and an element $x \in FC$ such that for every $D \in \mathbf{Ob}\,\mathcal{C}$ and $y \in FD$ there exists a unique morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ such that $y = (Ff)(x)$. The pair (C, x) is called a *universal element.*

b) Show that the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is representable. What is the universal element?

c) Let G be a group and $N < G$ be normal. Let

$$\begin{aligned} F : \mathbf{Gp} &\longrightarrow \mathbf{Set} \\ H &\longmapsto \{\phi : G \to H \mid \phi|_N \text{ is trivial}\}. \end{aligned}$$

Show that F is representable. What is the universal element? You should first determine what F does on morphisms.

3 Let $\mathcal{C}$ be a small category.

a) (Yoneda embedding) Using Yoneda Lemma, show that the functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \\ C &\longmapsto \mathrm{Hom}_{\mathcal{C}}(-, C) \\ f &\longmapsto f \circ - \end{aligned}$$

is fully faithful. Note that the category $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ is also denoted by $\mathrm{presh}_{\mathcal{D}}\mathcal{C}$, and called the category of *$\mathcal{D}$-valued presheaves on $\mathcal{C}$*.

b) Let $X, Y \in \mathbf{Ob}\mathcal{C}$. Show that $X \cong Y$ if and only if $\mathrm{Hom}_{\mathcal{C}}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, Y)$. In particular, a representation of a functor is unique up to isomorphisms.

4 A *preordered set* $(X, \leq)$ is a set with a binary relation $\leq$ on elements that is reflexive and transitive (it does not need to be antisymmetric).

a) Let $X = \{a, b, c\}$ with the preorder $a \leq b$, $b \leq a$, $b \leq c$. Let $Y = \{1, 2\}$ be the poset with partial order $1 \leq 2$. Show that the categories $\mathcal{C}_{(X,\leq)}$ and $\mathcal{C}_{(Y,\leq)}$ are equivalent.

b) Let X be a preordered set. Find a poset Y such that the categories $\mathcal{C}_{(X,\leq)}$ and $\mathcal{C}_{(Y,\leq)}$ are equivalent.