Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3204 Homological Algebra Fall 2018

Exercise set 11

1 Let $\mathscr{T}$ be a triangulated category and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle.

a) Show that $g \circ f = h \circ g = 0$.

b) Show that the following are equivalent:

1. f is a split monomorphism;
2. f is a monomorphism;
3. $h = 0$;
4. g is a split epimorphism;
5. g is an epimorphism.

c) Show that f is an isomorphism if and only if $Z = 0$.

2 Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

and

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$$

be distinguished triangles. Show that

$$X \oplus X' \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}} Z \oplus Z' \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & h' \end{pmatrix}} X[1] \oplus X'[1]$$

is also a distinguished triangle.

3 Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle and let $(0, \mu, 0)$ and $(0, \nu, 0)$ be two endomorphims of this triangle. Show that $\mu \circ \nu = 0$.

4 Let $\mathscr{A}$ be an abelian category. Let A^* be a complex concentrated in negative degrees ($A^n = 0\ \forall n \geq 0$) and B^* be a complex concentrated in non-negative degrees ($B^n = 0\ \forall n < 0$). Show that $\operatorname{Hom}_{\mathbf{D}(\mathscr{A})}(A^*, B^*) = 0$.

5 Let $\mathscr{A}$ be an abelian category. Assume that $\mathscr{A}$ is *hereditary*, that is, $\operatorname{Ext}^2_{\mathscr{A}}(A, B) = 0$ for all objects $A, B \in \mathscr{A}$. The goal of this exercise is to show that any complex X^* in $\mathbf{C}(\mathscr{A})$ is quasi-isomorphic to

$$H^*(X^*) := \cdots \to H^{-1}(X^*) \xrightarrow{0} H^0(X^*) \xrightarrow{0} H^1(X^*) \to \cdots ,$$

that is, any object in $\mathbf{D}(\mathscr{A})$ is isomorphic to its homology.

a) For every $n \in \mathbb{Z}$, show that there is an exact sequence

$$\operatorname{Ext}^1_{\mathscr{A}}(H^n(X^*), X^{n-1}) \to \operatorname{Ext}^1_{\mathscr{A}}(H^n(X^*), \operatorname{Im} d_X^{n-1}) \to 0.$$

b) For every $n \in \mathbb{Z}$, show that there exists $E^n \in \mathbf{Ob}\,\mathscr{A}$ and a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X^{n-1} & \longrightarrow & E^n & \longrightarrow & H^n(X^*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im} d_X^{n-1} & \longrightarrow & \ker d_X^n & \longrightarrow & H^n(X^*) & \longrightarrow & 0 \end{array}$$

c) Show that both X^* and $H^*(X^*)$ are quasi-isomorphic to a complex C^*, where $C^n = E^n \oplus X^n$.