Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 10

1 In $\mathbf{Mod}\, R$, consider two extensions

$$\mathbb{E}_1 : 0 \to B \xrightarrow{f_1} E_1 \xrightarrow{g_1} A \to 0 \quad \text{and} \quad \mathbb{E}_2 : 0 \to B \xrightarrow{f_2} E_2 \xrightarrow{g_2} A \to 0.$$

Let

$$X := \{(e, e') \in E_1 \oplus E_2 \mid g_1(e) = g_2(e')\}$$

and

$$Y := X/\{(f_1(b), 0) - (0, f_2(b)) \mid b \in B\}.$$

Show that the sequence

$$0 \to B \xrightarrow{b \mapsto [(f_1(b),0)]} Y \xrightarrow{[(e,e')] \mapsto g_1(e)} A \to 0$$

is well-defined and exact, and that it is the Baer sum of $\mathbb{E}_1$ and $\mathbb{E}_2$.

2 Calculate explicitly (i.e. by determining the equivalence classes of extensions) the following groups.

a) $\mathrm{YExt}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$;

b) $\mathrm{YExt}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

3 Let $\mathscr{A}$ be an abelian category with enough projectives. The *projective dimension* $\mathrm{pd}\, A$ of an object A is the smallest n, such that A has a projective resolution of the form

$$\cdots \to 0 \to P^{-n} \to \cdots \to P^0 \to 0 \to \cdots$$

Show that the following are equivalent:

1. $\mathrm{pd}\, A \le n$;
2. $\mathrm{Ext}^m_{\mathscr{A}}(A, B) = 0$ for all $m > n$ and all $B \in \mathbf{Ob}\, \mathscr{A}$;
3. $\mathrm{Ext}^{n+1}_{\mathscr{A}}(A, B) = 0$ for all $B \in \mathbf{Ob}\, \mathscr{A}$;
4. If $0 \to K^n \to P^{-(n-1)} \to \cdots \to P^0 \to A \to 0$ is an exact complex with P^{-i} projective, then the syzygy K^n is also projective.

4 Let $R = \mathbb{F}[X, Y]/(XY)$ for some field $\mathbb{F}$. Consider the double complex $X^{*,*}$ given by

$$X^{m,n} = R, \qquad d_h^{m,n} = d_v^{m,n} = \begin{cases} X & \text{if } m+n \text{ even} \\ Y & \text{if } m+n \text{ odd.} \end{cases}$$

Show that all rows and all columns of $X^{*,*}$ are exact, but $\mathrm{Tot}(X^{*,*})$ is not.

5 The goal of this exercise is to give another proof that Ext is independent of with respect to which argument we derive the Hom-functor. You may therefore not use this fact in the following. Let $\mathscr{A}$ be an abelian category with enough projectives.

a) 1. Let $F, G, H : \mathscr{A} \to \mathscr{B}$ be right exact contravariant functors between abelian categories. Show that if there exists natural transformations $F \Longrightarrow G \Longrightarrow H$ such that $0 \to FP \to GP \to HP \to 0$ is a short exact sequence for any projective P, then for any $X \in \mathbf{Ob}\,\mathscr{A}$ there is a long exact sequence

$$0 \to FX \to GX \to HX \to \mathbb{R}^1 FX \to \mathbb{R}^1 GX \to \mathbb{R}^1 HX \to \mathbb{R}^2 FX \to \dots$$

2. Conclude that for any short exact sequence $0 \to A \to B \to C \to 0$ and any object X in $\mathscr{A}$, there is a long exact sequence

$$\begin{aligned} \dots &\to \mathrm{Ext}^n_{\mathscr{A}}(-, A)(X) \to \mathrm{Ext}^n_{\mathscr{A}}(-, B)(X) \to \mathrm{Ext}^n_{\mathscr{A}}(-, C)(X) \\ &\to \mathrm{Ext}^{n+1}_{\mathscr{A}}(-, A)(X) \to \mathrm{Ext}^{n+1}_{\mathscr{A}}(-, B)(X) \to \mathrm{Ext}^{n+1}_{\mathscr{A}}(-, C)(X) \to \dots \end{aligned}$$

b) Let I be an injective object. Show that $\mathrm{Ext}^n_{\mathscr{A}}(-, I)(X) = 0$ for all $n > 0$.

c) Show that, given a short exact sequence

$$0 \to A \to I \to \mho A \to 0$$

with I injective, we have

$$\mathrm{Ext}^n_{\mathscr{A}}(-, A)(X) \cong \begin{cases} \mathrm{Ext}^{n-1}_{\mathscr{A}}(-, \mho A)(X) & \text{if } n \geq 2 \\ \operatorname{cok}\left[\mathrm{Hom}_{\mathscr{A}}(X, I) \to \mathrm{Hom}_{\mathscr{A}}(X, \mho A)\right] & \text{if } n = 1 \end{cases}$$

Remark: In a) and c), be careful that although the theorems seem familiar, they are not what we proved in class. They are rather a version in the "inner" argument.

d) Assume that $\mathscr{A}$ in addition has enough injectives. Show by induction on n that

$$\mathrm{Ext}^n_{\mathscr{A}}(-, A)(X) \cong \mathrm{Ext}^n_{\mathscr{A}}(X, -)(A).$$