Norwegian University of Science and Technology
Department of Mathematical Sciences

MA3204 Homological Algebra Fall 2018

Exercise set 1

1 Show that

a) Any split monomorphism is a monomorphism.

b) The following are equivalent, for a morphism f:

- f is an isomorphism;
- f is a split monomorphism and an epimorphism;
- f is a monomorphism and a split epimorphism.

2 **a)** Consider the category **Ring** of rings with unit. Show that the inclusion

$$\iota : \mathbb{Z} \to \mathbb{Q}$$

is both a monomorphism and an epimorphism, but not an isomorphism. More generally, this is true for any inclusion $\iota : D \to S^{-1}D$ from an integral domain D to its localization at a non-trivial multiplicatively closed set S. This shows that an epimorphism is not necessarily surjective.

b) Consider the category **Ab** of abelian groups. Show that the inclusion

$$\iota : \mathbb{Z} \to \mathbb{Q}$$

is not an epimorphism.

c) Show that in the category **Ab**, a morphism is a monomorphism if and only if it is injective, and that it is an epimorphism if and only if it is surjective. Then the isomorphisms are exactly the monomorphisms/epimorphisms.

3 Determine whether the following functors are full, faithful or dense.

a) The inclusion functor: $\mathbf{Ab} \to \mathbf{Gp}$;

b) The functor which forgets the topology: $\mathbf{Top} \to \mathbf{Set}$ from the category of topological spaces to the category of sets;

c) The Hom-functor $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, -) : \mathbf{Ab} \to \mathbf{Set}$.

4 Let $\mathcal{C}$ be a small category (class of objects form a set) and $\mathcal{D}$ a category. We denote by $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ the collection with objects given by all functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms given by all natural transformations between them. Show that $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is a category, called the *functor category*.