Problem set 6

Problem 1. Show the following for a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in a triangulated category.

(1) f is a split monomorphism $\iff$ g is a split epimorphism $\iff$ $h = 0$;
(2) f is an isomorphism $\iff$ $Z = 0$.

Problem 2. Let $\mathcal{T}$ be a triangulated category. Show that if $\mathcal{T}$ is moreover abelian, then $\mathcal{T}$ is semisimple.

Problem 3. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

and

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1]$$

be distinguished triangles in a triangulated category. Show that

$$X \oplus X' \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & f' \end{smallmatrix}\right)} Y \oplus Y' \xrightarrow{\left(\begin{smallmatrix} g & 0 \\ 0 & g' \end{smallmatrix}\right)} Z \oplus Z' \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & h' \end{smallmatrix}\right)} X[1] \oplus X'[1]$$

is also a distinguished triangle.

Problem 4 (Challenge). Let $\mathcal{A}$ be a hereditary abelian category. Show that any object in $\mathsf{D}(\mathcal{A})$ is isomorphic to its homology. That is, given a complex A, show that it is quasi-isomorphic to

$$\cdots \longrightarrow \mathsf{H}^{-1}(A) \xrightarrow{0} \mathsf{H}^{0}(A) \xrightarrow{0} \mathsf{H}^{1}(A) \longrightarrow \cdots .$$