Problem set 5

Problem 1. For any ring R, $\mathsf{mod}\, R$ denotes the category of finitely generated R-modules. Let $\mathcal{T} \subset \mathsf{Mod}\,\mathbb{Z}$ be the category of all torsion abelian groups.

(1) Show that $\mathsf{mod}\,\mathbb{Z}$ is an abelian category with enough projectives, but not enough injectives.
(2) Show that $\mathcal{T}$ is an abelian category with enough injectives, but not enough projectives.

Let $\mathcal{F} \subset \mathsf{Mod}\,\mathbb{Z}$ be the category of all torsion-free abelian groups.

(3) Show that $\mathcal{F}$ is not an abelian category.
(4) Find a ring R for which $\mathsf{mod}\, R$ is not abelian.

Problem 2 (Generalized Schanuel's Lemma). Let R be a ring. Given exact sequences

$$0 \longrightarrow K \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow A \longrightarrow 0$$

in $\mathsf{Mod}\, R$ with each P_i and each Q_i projective, show that

$$K \oplus Q_n \oplus P_{n-1} \oplus \cdots \cong L \oplus P_n \oplus Q_{n-1} \oplus \cdots .$$

Problem 3. Let R be a ring and $M \in \mathsf{Mod}\, R$. Show that the following are equivalent.

(1) $\mathsf{pd}\, M \leq n$;
(2) $\mathsf{Ext}^k_R(M, N) = 0$ for each $N \in \mathsf{Mod}\, R$ and $k \geq n+1$;
(3) $\mathsf{Ext}^{n+1}_R(M, N) = 0$ for each $N \in \mathsf{Mod}\, R$;
(4) there is a projective resolution of M with projective $(n-1)$st syzygy;
(5) each projective resolution of M has projective $(n-1)$st syzygy.

Problem 4. Let $\mathbb{F}$ be a field and $R = \mathbb{F}[x, y]$. Consider the R-module $R/(x, y) = \mathbb{F}$. Show that

$$\mathsf{Ext}^i_R(\mathbb{F}, \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } i = 0 \\ \mathbb{F}^2 & \text{if } i = 1 \\ \mathbb{F} & \text{if } i = 2 \\ 0 & \text{if } i \geq 3. \end{cases}$$

Problem 5. Let $\mathbb{F}$ be a field. Calculate $\mathsf{Ext}^i_{\mathsf{presh}_{\mathsf{mod}\,\mathbb{F}}\, X}(I_\omega, P_0)$ for the poset

(1) $X = \{0 < \omega\}$;
(2) $X = \left\{ \begin{array}{ccc} & \omega & \\ a & & b \\ & 0 & \end{array} \right\}$.

Problem 6. Let R be a ring over which each projective module is injective. Show that $\mathsf{gldim}\, R$ is either 0 or ∞.

Problem 7. Calculate explicitly (i.e. by determining the equivalence classes of extensions) the groups

(1) $\mathsf{YExt}^1_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(3))$;
(2) $\mathsf{YExt}^1_{\mathbb{Z}}(\mathbb{Z}/(2), \mathbb{Z}/(2))$.

Problem 8. Let $\mathcal{A}$ be a semisimple abelian category. Show that any object in $\mathsf{K}(\mathcal{A})$ is isomorphic to its homology. That is, given a complex A, show that

$$A \simeq \cdots \longrightarrow \mathsf{H}^{-1}(A) \xrightarrow{0} \mathsf{H}^0(A) \xrightarrow{0} \mathsf{H}^1(A) \longrightarrow \cdots .$$