Problem set 4

Problem 1. Let R be a ring, A an R-module and (B_i) a family of R-modules. Show that

(1) $\mathsf{Hom}_R\left(\coprod_i B_i, A\right) \cong \prod_i \mathsf{Hom}_R\left(B_i, A\right)$;
(2) $\mathsf{Hom}_R\left(A, \prod_i B_i\right) \cong \prod_i \mathsf{Hom}_R\left(A, B_i\right)$;
(3) if A is finitely generated, then $\mathsf{Hom}_R\left(A, \coprod_i B_i\right) \cong \coprod_i \mathsf{Hom}_R\left(A, B_i\right)$.

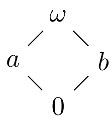
Suppose now that (B_i) is a family of R^{op}-modules. Show that

(4) $A \otimes_R \left(\coprod_i B_i\right) \cong \coprod_i \left(A \otimes_R B_i\right)$.

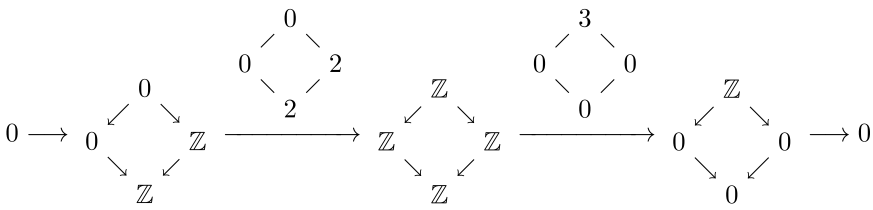
Provide examples where

(5) $\mathsf{Hom}_R\left(A, \coprod_i B_i\right) \ncong \coprod_i \mathsf{Hom}_R\left(A, B_i\right)$;
(6) $\mathsf{Hom}_R\left(\prod_i B_i, A\right) \ncong \coprod_i \mathsf{Hom}_R\left(B_i, A\right)$;
(7) $A \otimes_R \left(\prod B_i\right) \ncong \prod \left(A \otimes_R B_i\right)$.

Problem 2. Let X be the poset



and consider the following complex of $\mathsf{Mod}\,\mathbb{Z}$-valued presheaves on X.



Calculate the homologies of this complex.

Problem 3. Let $\mathbb{F}$ be a field. Find a projective resolution of I_ω in $\mathsf{presh}_{\mathsf{mod}\,\mathbb{F}}\,X$ when

(1) X is the poset $\{0 < \omega\}$;
(2) X is the poset of Problem 2 above.

Problem 4 (Schanuel's Lemma). Let R be a ring. Given short exact sequences

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow A \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow K_2 \longrightarrow P_2 \longrightarrow A \longrightarrow 0$$

in $\mathsf{Mod}\,R$ with P_1 and P_2 projective, show that

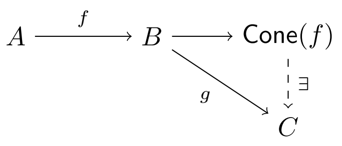
$$K_1 \oplus P_2 \cong K_2 \oplus P_1.$$

Problem 5 (Exactness of colim). Let X be a 'directed' poset, i.e.

$$\forall\, x, y \in X\ \exists\ z \in X \text{ such that } x \leq z \text{ and } y \leq z.$$

Show that, for any ring R, the functor $\mathsf{colim}\colon \mathsf{presh}_{\mathsf{Mod}\,R}\,X \longrightarrow \mathsf{Mod}\,R$ is exact. (Hint: We have already seen that this is a right exact functor.)

Problem 6 (Cones are weak cokernels in $\mathbf{K}(\mathcal{A})$). Let $\mathcal{A}$ be an abelian category and let $f\colon A \longrightarrow B$ be a morphism in $\mathbf{C}(\mathcal{A})$. Suppose that $g\colon B \longrightarrow C$ is a morphism such that $gf = 0$ in $\mathbf{K}(\mathcal{A})$. Show that g factors through the canonical $B \longrightarrow \mathsf{Cone}(f)$:



Provide an example where the factorization is not unique.

Problem 7. Let $\mathcal{A}$ be an abelian category and let $A \in \mathbf{C}(\mathcal{A})$ be such that $\mathsf{H}^i(A) = 0$ for each $i < 0$. Show that there is a quasi-isomorphism $A \longrightarrow B$ for some $B \in \mathbf{C}(\mathcal{A})$ with $B^i = 0$ for each $i < 0$.

Problem 8. Let $\mathcal{A}$ be an abelian category and $A \in \mathbf{C}(\mathcal{A})$. Show that $A \cong 0$ in $\mathbf{K}(\mathcal{A})$ if and only if A is isomorphic to a complex of the form

$$\cdots \longrightarrow X^{-1} \oplus X^0 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} X^0 \oplus X^1 \xrightarrow{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} X^1 \oplus X^2 \longrightarrow \cdots .$$

(Hint: For the non-trivial direction, consider the short exact sequence

$$0 \longrightarrow \mathsf{B}^n(A) \longrightarrow A^n \longrightarrow \mathsf{B}^{n+1}(A) \longrightarrow 0.)$$

Problem 9 (Challenge). The goal of this exercise is to extablish the following.

Claim. Let R be any ring. If $B \in \mathsf{Mod}\, R$ is such that $0 \longrightarrow B \otimes_R I \longrightarrow B \otimes_R R$ is exact for each finitely generated left ideal I of R, then B is flat.

Show the claim by breaking the proof into the following steps.

Colimits: If $0 \longrightarrow B \otimes_R I \longrightarrow B \otimes_R R$ is exact for each finitely generated left ideal I, then $0 \longrightarrow B \otimes_R I' \longrightarrow B \otimes_R R$ is exact for *any* left ideal I'. (Hint: Problem 5)

Character modules: The *character module* $B^* = \mathsf{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) \in \mathsf{Mod}\, R^{\mathsf{op}}$ of B is injective if and only if B is flat. (Hint: show that

(1) $\mathbb{Q}/\mathbb{Z}$ is a 'cogenerator' of $\mathsf{Mod}\, \mathbb{Z}$ (i.e. for each $M \in \mathsf{Mod}\, \mathbb{Z}$ and each $0 \neq m \in M$, there is a morphism $f \colon M \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $f(m) \neq 0$);

(2) a sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathsf{Mod}\, R$ is exact if and only if the sequence $0 \longrightarrow Z^* \longrightarrow Y^* \longrightarrow X^* \longrightarrow 0$ is exact in $\mathsf{Mod}\, R^{\mathsf{op}}$;

(3) for a ring S, a module ${}_S X_R$ which is flat over R and an injective module ${}_S Y$, $\mathsf{Hom}_S(X, Y)$ becomes an injective left R-module.)

Wrap it up: Conclude that the claim holds using Baer's Criterion.