Problem set 3

Problem 1 (Additive functors). Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between additive categories. Show that F is additive if and only if it preserves biproducts.

Problem 2 (Split exact sequences). Let R be a ring. A short exact sequence

$$(*) \qquad 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathsf{Mod}\, R$ is called *split exact* if g is a split epimorphism. Show that the following are equivalent for the short exact sequence $(*)$.

(1) $(*)$ is split exact;
(2) f is a split monomorphism;
(3) id_B can be written as $fk + hg$.

Show that if $(*)$ is split exact, then $B \cong A \oplus C$. Show that the converse does not hold. (Hint: Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/(2) \longrightarrow 0$$

and add

$$0 \longrightarrow 0 \longrightarrow \coprod_{\mathbb{Z}} \mathbb{Z}/(2) \xrightarrow{\mathsf{id}} \coprod_{\mathbb{Z}} \mathbb{Z}/(2) \longrightarrow 0$$

to it.)

Problem 3. Let $(F, G)\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an adjoint pair between abelian categories. Show that

(1) F is right exact and G is left exact;
(2) G right exact $\Longrightarrow$ F preserves projective objects;
(3) F left exact $\Longrightarrow$ G preserves injective objects.

Problem 4. The aim of this exercise is to construct projective and injective presheaves.

Let X be a poset, $\mathbb{F}$ a field and $\mathsf{mod}\, \mathbb{F}$ the category of finite dimensional vector spaces over $\mathbb{F}$. Recall that the category $\mathsf{presh}_{\mathsf{mod}\,\mathbb{F}} X$ of presheaves on X with values in $\mathsf{mod}\, \mathbb{F}$ is abelian — kernels and cokernels being constructed pointwise.

For $i \in X$, consider the presheaves P_i and I_i given by

$$P_i(j) = \begin{cases} \mathbb{F} & \text{if } j \leq i \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad I_i(j) = \begin{cases} \mathbb{F} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

The inclusion $\iota\colon \{i\} \hookrightarrow X$ induces a functor

$$\iota^*\colon \mathsf{presh}_{\mathsf{mod}\,\mathbb{F}} X \longrightarrow \mathsf{presh}_{\mathsf{mod}\,\mathbb{F}} \{i\} = \mathsf{mod}\, \mathbb{F}$$

given by $M \mapsto M \circ \iota$. Moreover, ι^* admits a right adjoint R and a left adjoint L. Convince yourself that

(1) ι^* is exact;
(2) $L(\mathbb{F}) = P_i$;
(3) $R(\mathbb{F}) = I_i$.

Conclude that P_i is a projective presheaf and that I_i is an injective presheaf.

Finally, let $X = \{a > 0 < b\}$. Find a projective object P and an epimorphism $P \longrightarrow I_0$ in $\mathsf{presh}_{\mathsf{mod}\,\mathbb{F}} X$.

Problem 5. (1) Let G be a finite abelian group. What is $G \otimes_{\mathbb{Z}} \mathbb{Q}$?
(2) Show that $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong \mathbb{Z}/(\mathsf{gcd}(m, n))$.
(3) What is $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$?

Problem 6. Let R be a commutative ring, and let P and Q be projective R-modules. Show that $P \otimes_R Q$ is a projective R-module.

Problem 7. Let $\mathcal{A}$ be an abelian category. Show that

(1) $P \in \mathcal{A}$ is projective $\iff$ any short exact sequence ending at P is split;
(2) $I \in \mathcal{A}$ is injective $\iff$ any short exact sequence starting at I is split.

Give different (simpler) proofs of the implications '$\Longleftarrow$' in the case $\mathcal{A} = \mathsf{Mod}\, R$. (Hint: Any module admits an epimorphism from a projective module and a monomorphism into an injective module.)

Problem 8. Let R be a ring. In the lectures we have outlined a proof of the fact that any R-module admits a monomorphism into an injective R-module. The goal of this exercise is to provide the missing details.

Baer's Criterion: Show that $E \in \mathsf{Mod}\, R$ is injective $\iff$ for each right ideal $I \subset R$, each R-linear map $I \longrightarrow E$ extends to an R-linear map $R \longrightarrow E$. (Hint: For the non-trivial implication, take an inclusion

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \hookrightarrow & B \\ & & \downarrow{\scriptstyle g} & & \\ & & E & & \end{array}$$

and apply Zorn's Lemma to the set

$$\mathcal{S} = \{(A', g') \mid A \subset A' \subset B,\ A' \xrightarrow{g'} E \text{ extends } g\}.)$$

Divisibility: Check that

(1) over a principal ideal domain, any divisible module is injective;
(2) a coproduct of divisible modules is divisible;
(3) a quotient of a divisible module is divisible.

Injective group $\rightsquigarrow$ injective R-module: Let D be an injective $\mathbb{Z}$-module. Show that $\mathsf{Hom}_{\mathbb{Z}}(R, D)$ becomes an injective R-module. (Hint: Hom has a left adjoint.)