Problem set 1

Problem 1. Let $f\colon X \longrightarrow Y$ be a morphism of $\mathcal{C}$. Show that

(1) if f is a split monomorphism, then it is a monomorphism; and
(2) if f is both a split monomorphism and an epimorphism, then it is an isomorphism.

Let $\iota\colon \mathbb{Z} \longrightarrow \mathbb{Q}$ be the inclusion in the category **Rings** (consisting of associative rings with unit, whose morphisms are ring homomorphisms preserving the units). Show that ι is both a monomorphism and an epimorphism, but not an isomorphism.

Problem 2. Exercises I.12 and I.14 in the book.

Problem 3. Recall that $(F, G)\colon \mathcal{C} \longrightarrow \mathcal{D}$ is an adjoint pair if there exists a natural isomorphism $\varphi\colon \mathsf{Hom}_{\mathcal{D}}(F-, -) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(-, G-)$ as functors $\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \longrightarrow \mathbf{Set}$, i.e., there are bijections

$$\varphi_{X,Y}\colon \mathsf{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(X, GY),$$

natural in both $X \in \mathcal{C}$ and $Y \in \mathcal{D}$.

Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be an equivalence of categories with quasi-inverse G. Show that both (F, G) and (G, F) are adjoint pairs of functors.

Now let (F, G) be an arbitrary adjoint pair of functors. The aim is to show that F and G give rise to natural transformations that verify the so-called 'triangle identities'.

(1) For every X in $\mathcal{C}$, define $\eta_X\colon X \longrightarrow GFX$ as $\varphi_{X,FX}(\mathsf{id}_{FX})$. Show that these components give rise to a natural transformation (the unit of the adjunction) $\eta\colon \mathsf{id}_{\mathcal{C}} \longrightarrow GF$. (Hint: It might be helpful to look at the following rather large diagram, which commutes for every $f\colon X \longrightarrow X'$ in $\mathcal{C}$, by the naturality assumption on φ,

$$\begin{array}{ccc}
\mathsf{Hom}_{\mathcal{D}}(FX, FX) & \xrightarrow{\quad \varphi_{X,FX} \quad} & \mathsf{Hom}_{\mathcal{C}}(X, GFX) \\
\big\downarrow{\scriptstyle (Ff)_*} & & \big\downarrow{\scriptstyle (GFf)_*} \\
\mathsf{Hom}_{\mathcal{D}}(FX, FX') & \xrightarrow{\quad \varphi_{X,FX'} \quad} & \mathsf{Hom}_{\mathcal{C}}(X, GFX') \\
\big\uparrow{\scriptstyle (Ff)^*} & & \big\uparrow{\scriptstyle f^*} \\
\mathsf{Hom}_{\mathcal{D}}(FX', FX') & \xrightarrow{\quad \varphi_{X',FX'} \quad} & \mathsf{Hom}_{\mathcal{C}}(X', GFX')
\end{array}$$

by tracing the identities on FX and FX' around.) Dually, one obtains the counit $\varepsilon\colon FG \longrightarrow \mathsf{id}_{\mathcal{D}}$ at $Y \in \mathcal{D}$ as $\varepsilon_Y = \varphi^{-1}_{GY,Y}(\mathsf{id}_{GY})$.
(2) Use naturality of η and ε to show that if $f\colon FX \longrightarrow Y$, then $\varphi_{X,Y}(f) = Gf \circ \eta_X$; and if $g\colon X \longrightarrow GY$, then $\varphi^{-1}_{X,Y}(g) = \varepsilon_Y \circ Fg$.
(3) Use the previous point to conclude that for every $X \in \mathcal{C}$, $\varepsilon_{FX} \circ F(\eta_X) = \mathsf{id}_{FX}$, and for every $Y \in \mathcal{D}$, $G(\varepsilon_Y) \circ \eta_{GY} = \mathsf{id}_{GY}$, so that as natural transformations, the compositions

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon_{F-}} F; \qquad G \xrightarrow{\eta_{G-}} GFG \xrightarrow{G\varepsilon} G$$

are the identity natural transformations on F and G respectively.

Problem 4. Let **Rng** denote the category of associative non-unital rings, whose morphisms are ring homomorphisms. There is a forgetful functor $\mathtt{f}\colon \mathbf{Ring} \longrightarrow \mathbf{Rng}$.

(1) Argue that $\mathtt{f}$ is faithful, but neither full nor dense.

(2) Let $R \in \mathbf{Rng}$. Let R_+ have underlying set $R \times \mathbb{Z}$, and define addition and multiplication as $(r, n) + (r', n') = (r + r', n + n')$ and $(r, n) \cdot (r', n') = (rr' + rn' + r'n, nn')$. Show that this makes R_+ into a unital ring.

(3) Show that $j \colon R \longrightarrow R_+$, $r \mapsto (r, 0)$, is a ring homomorphism satisfying the following universal property: For each unital ring S and ring homomorphism $f \colon R \longrightarrow S$ there exists a unique unital ring homomorphism $\bar{f} \colon R_+ \longrightarrow S$ such that $\bar{f}j = f$.

(4) Show that $-_+ \colon \mathbf{Rng} \longrightarrow \mathbf{Ring}$ is a functor, which is furthermore left adjoint to $\mathtt{f}$. Determine the unit η_R and the counit ε_R of the adjunction.