MA3204: HOMOLOGICAL ALGEBRA - EXERCISE SHEET 3

Exercise 1. (The Yoneda embedding reflects exactness) Let $\mathscr{A}$ be an abelian category and $X \xrightarrow{f} Y \xrightarrow{g} Z$ a sequence in $\mathscr{A}$. Consider the following statements:

(i) The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact.

(ii) For any object A in $\mathscr{A}$, the sequence

$$\mathsf{Hom}_{\mathscr{A}}(A, X) \xrightarrow{f_*} \mathsf{Hom}_{\mathscr{A}}(A, Y) \xrightarrow{g_*} \mathsf{Hom}_{\mathscr{A}}(A, Z)$$

is exact.

(iii) For any object A in $\mathscr{A}$, the sequence

$$\mathsf{Hom}_{\mathscr{A}}(Z, A) \xrightarrow{g^*} \mathsf{Hom}_{\mathscr{A}}(Y, A) \xrightarrow{f^*} \mathsf{Hom}_{\mathscr{A}}(X, A)$$

is exact.

Show that (ii)$\Longrightarrow$(i) and (iii)$\Longrightarrow$(i).

Exercise 2. Let (F, G) be an adjoint pair of functors between the abelian categories $\mathscr{A}$ and $\mathscr{B}$.

(i) Show that F is right exact and G is left exact.

(ii) Assume that the functor G is right exact. Show that the functor F preserves projective objects.

(iii) Assume that the functor F is left exact. Show that the functor G preserves injective objects.

Exercise 3. Let $\phi\colon R \longrightarrow S$ be a homomorphism of rings. Show that $(\phi^*, \phi_*, \phi^!)$:

$$\begin{tikzcd} \mathsf{Mod}\text{-}S \arrow[rr, "\phi_*"] & & \mathsf{Mod}\text{-}R \arrow[ll, bend right, "\phi^*"'] \arrow[ll, bend left, "\phi^!"] \end{tikzcd}$$

is an adjoint triple, i.e. (ϕ^*, ϕ_*) and $(\phi_*, \phi^!)$ are adjoint pairs, where ϕ_* is restriction of scalars, $\phi^*(M) = M \otimes_R S$ is extension of scalars, and $\phi^!(M) = \mathsf{Hom}_R(S, M)$.

Exercise 4. Let R be a ring and $e^2 = e$ and idempotent element of R.

(i) Show that the triple of functors $(Re \otimes_{eRe} -, e(-), \mathsf{Hom}_{eRe}(eR, -))$:

$$\begin{tikzcd} \mathsf{Mod}\text{-}R \arrow[rr, "e(-)"] & & \mathsf{Mod}\text{-}eRe \arrow[ll, bend right, "Re\otimes_{eRe}-"'] \arrow[ll, bend left, "\mathsf{Hom}_{eRe}(eR,-)"] \end{tikzcd}$$

is an adjoint triple, where $e(-)$ is multiplication with the idempotent element e.

(ii) Show that the functor $Re\otimes_{eRe}-\colon \mathsf{Mod}\text{-}eRe \longrightarrow \mathsf{Mod}\text{-}R$ is fully faithful. Is the functor $\mathsf{Hom}_{eRe}(eR, -)$ fully faithful ?

(iii) What is the kernel of the functor $e(-)\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}eRe$?

Exercise 5. Let R be a commutative ring and I, J two ideals in R. Show that $R/I \otimes_R R/J \cong R/(I + J)$. Using this, show that $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$ where $d = \mathsf{gcd}(m, n)$.

Exercise 6. Let R and S be two k-algebras where k is a commutative ring. Show that $R \otimes_k S$ is a k-algebra with multiplication: $(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'$.

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Exercise 7. Show that the k-algebras $k[x, y]$ and $k[x] \otimes_k k[y]$ are isomorphic.

Exercise 8. If R is a k-algebra, show that the k-algebras $R \otimes_k \mathsf{M}_n(k)$ and $\mathsf{M}_n(R)$ are isomorphic.

Exercise 9. Let R be a ring, ${}_RA$ an R-module and $(B_i)_{i \in I}$ a family of left R-modules.

(i) Show that we have the following isomorphisms:

$$\mathsf{Hom}_R(A, \prod_{i \in I} B_i) \cong \prod_{i \in I} \mathsf{Hom}_R(A, B_i)$$

and

$$\mathsf{Hom}_R(\coprod_{i \in I} B_i, A) \cong \prod_{i \in I} \mathsf{Hom}_R(B_i, A)$$

(ii) Provide examples where the following groups are not isomorphic:

(a) $\mathsf{Hom}_R(A, \coprod_{i \in I} B_i) \ncong \prod_{i \in I} \mathsf{Hom}_R(A, B_i)$.

(b) $\mathsf{Hom}_R(A, \coprod_{i \in I} B_i) \ncong \coprod_{i \in I} \mathsf{Hom}_R(A, B_i)$.

(c) $\mathsf{Hom}_R(\prod_{i \in I} B_i, A) \ncong \prod_{i \in I} \mathsf{Hom}_R(B_i, A)$.

(d) $\mathsf{Hom}_R(\prod_{i \in I} B_i, A) \ncong \coprod_{i \in I} \mathsf{Hom}_R(B_i, A)$.

(iii) Show that if ${}_RA$ is finitely generated, then there is an isomorphism:

$$\mathsf{Hom}_R(A, \coprod_{i \in I} B_i) \cong \coprod_{i \in I} \mathsf{Hom}_R(A, B_i)$$

Exercise 10. Exercises III.1 – III.4 from the notes.

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