MA3204: HOMOLOGICAL ALGEBRA - EXERCISE SHEET 2

Exercise 1. Let $\mathcal{X}$ be a small category and $\mathscr{A}$ an abelian category. Show that the category of functors $\mathsf{Fun}(\mathcal{X}, \mathscr{A})$ is abelian.

Exercise 2. A functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between abelian categories is additive, if $F(f + f') = F(f) + F(f')$ for any morphisms $f, f'\colon X \longrightarrow Y$ in $\mathscr{A}$. Equivalently, the map $\mathsf{Hom}_{\mathscr{A}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathscr{B}}(FX, FY)$, $f \mapsto F(f)$, is a group homomorphism. Show that the following are equivalent:

(i) The functor F is additive.
(ii) F preserves finite products.
(iii) F preserves finite coproducts.

Exercise 3. Let R be a ring. Show that there are the following equivalences:

(i) $\mathsf{Mod}\text{-}R^{\mathsf{op}} \xrightarrow{\simeq} \mathsf{Fun}(R, \mathscr{A}b)$.
(ii) $\mathsf{Mod}\text{-}R \xrightarrow{\simeq} \mathsf{Fun}(R^{\mathsf{op}}, \mathscr{A}b)$.

Note that in the above two functor categories we consider additive functors.
(Hint: Consider the ring R as a category with one object.)

Exercise 4. Let R be a ring. Consider the full subcategory $\mathsf{mod}\text{-}R$ of $\mathsf{Mod}\text{-}R$ consisting of all finitely generated R-modules. Show that:

(i) $\mathsf{mod}\text{-}R$ has cokernels, and
(ii) $\mathsf{mod}\text{-}R$ has kernels if and only if R is right Noetherian.

Exercise 5. (Challenge!) Let R be a ring. Recall that a ring R is called left coherent if every finitely generated left ideal in R is finitely presented. We denote[1] again by $\mathsf{mod}\text{-}R^{\mathsf{op}}$ the full subcategory of $\mathsf{Mod}\text{-}R^{\mathsf{op}}$ consisting of the finitely presented left R-modules. Show that:

$$\mathsf{mod}\text{-}R^{\mathsf{op}} \text{ is abelian} \Longleftrightarrow R \text{ is left coherent.}$$

Exercise 6. Let $\mathscr{A}$ be an additive category. A complex over $\mathscr{A}$ is a family $A^\bullet = (A^n, d^n)_{n\in\mathbb{Z}}$ where A^n are objects in $\mathscr{A}$ and $d_A^n\colon A^n \longrightarrow A^{n+1}$ are morphisms such that $d_A^n \circ d_A^{n-1} = 0$ for all $n \in \mathbb{Z}$. A complex is written as follows:

$$\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \xrightarrow{d^2} \cdots$$

A morphism of complexes $f^\bullet\colon A^\bullet \longrightarrow B^\bullet$ is a family of morphisms $f^\bullet = (f^n\colon A^n \longrightarrow B^n)$ such that $d_B^n \circ f^n = f^{n+1} \circ d_A^n$ for all $n \in \mathbb{Z}$, that is, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & A^{-1} \arrow[r, "d^{-1}"] \arrow[d, "f^{-1}"] & A^0 \arrow[r, "d^0"] \arrow[d, "f^0"] & A^1 \arrow[r, "d^1"] \arrow[d, "f^1"] & A^2 \arrow[r, "d^2"] \arrow[d, "f^2"] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & B^{-1} \arrow[r, "d^{-1}"] & B^0 \arrow[r, "d^0"] & B^1 \arrow[r, "d^1"] & B^2 \arrow[r, "d^2"] & \cdots
\end{tikzcd}$$

The complexes over $\mathscr{A}$ together with the morphisms of complexes form a category, which is called the category of complexes over $\mathscr{A}$ and is denoted by $\mathsf{C}(\mathscr{A})$.

Show that $\mathsf{C}(\mathscr{A})$ is an additive category. If $\mathscr{A}$ is an abelian category show that $\mathsf{C}(\mathscr{A})$ is also abelian.

Exercise 7. Exercises II.1 – II.9 from the notes.

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[1] Over "good" rings, these two types of modules is the same, see Rotman's book.