MA3204: HOMOLOGICAL ALGEBRA - EXERCISE SHEET 1

Exercise 1. Let $\mathscr{C}$ be a category. Show that $\mathscr{C}^{\mathsf{op}}$ is a category, called the opposite category.

Exercise 2. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. Show that $\mathscr{C} \times \mathscr{D}$ is a category, called the product category.

Exercise 3. Let R be a ring. Show that a morphism $f\colon X \longrightarrow Y$ in $\mathsf{Mod}\text{-}R$ is a monomorphism (respectively an epimorphism) if and only if the map f between the underlying sets is injective (respectively surjective). Show that the same characterisation holds for a map $f\colon G \longrightarrow H$ in the category Grp of groups.

Exercise 4. Show that a morphism in Ring is a monomorphism if and only if it it injective. Show with an example that epimorphisms in Ring need not be surjective.

Exercise 5. Let R be a ring and consider the functor $(-)^* := \mathsf{Hom}_R(-, R)\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Mod}\text{-}R^{\mathsf{op}}$. Show that the evaluation map

$$\mathsf{ev}\colon \mathrm{Id}_{\mathsf{Mod}\text{-}R} \longrightarrow (-)^{**},\ M \mapsto \mathsf{Hom}_R(\mathsf{Hom}_R(M, R), R)$$

is a natural transformation. Assume that $R = \mathbb{K}$ is a field. Prove that the evaluation map

$$\mathsf{ev}\colon \mathrm{Id}_{\mathsf{mod}\text{-}R} \longrightarrow (-)^{**}$$

is a natural isomorphism, where $\mathsf{mod}\text{-}R$ is now the category of finite dimensional $\mathbb{K}$-vector spaces. In this case, observe that the categories $\mathsf{mod}\text{-}R$ and $\mathsf{mod}\text{-}R^{\mathsf{op}}$ are equivalent.

Exercise 6. Let $\mathcal{X}$ be a small category (i.e. $\mathsf{Ob}\mathcal{X}$ form a set) and $\mathscr{C}$ a category. We denote by $\mathsf{Fun}(\mathcal{X}, \mathscr{C})$ the collection of all functors from $\mathcal{X}$ to $\mathscr{C}$ (objects) and natural transformations between them (morphisms). Show that $\mathsf{Fun}(\mathcal{X}, \mathscr{C})$ is a category, called the functor category.

Exercise 7. Let R be a ring and consider the group $\mathsf{U}(R)$ of invertible elements of R. Show that

$$\mathsf{U}\colon \mathsf{Ring} \longrightarrow \mathsf{Grp},\ R \mapsto \mathsf{U}(R)$$

is a functor.

Exercise 8. Let $\mathbb{K}$ be a field. Denote by $\mathsf{Mat}_{\mathbb{K}}$ the category with objects the natural numbers and $\mathsf{Hom}_{\mathsf{Mat}_{\mathbb{K}}}(m, n)$ being $m \times n$ matrices over $\mathbb{K}$. Show that the categories $\mathsf{Mat}_{\mathbb{K}}$ and $\mathsf{mod}\text{-}\mathbb{K}$ are equivalent.

Exercise 9. Let R be a ring and consider the ring $\mathsf{M}_n(R)$ of $n \times n$ matrices with entries in R. Show that the categories of modules $\mathsf{Mod}\text{-}R$ and $\mathsf{Mod}\text{-}\mathsf{M}_n(R)$ are equivalent.

Exercise 10. Assume that F and F' are left adjoints of a functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$. Show that F and F' are naturally isomorphic.

Exercise 11. Let (F, G) be an adjoint pair between the categories $\mathscr{C}$ and $\mathscr{D}$.

(i) Show that the functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ is faithful if and only if the counit $\varepsilon_D\colon FG(D) \longrightarrow D$ is an epimorphism.
(ii) Show that the functor $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ is full if and only if the counit $\varepsilon_D\colon FG(D) \longrightarrow D$ is a split monomorphism.
(iii) Show that G is fully faithful if and only if the counit ε_D is an isomorphism for all D in $\mathscr{D}$.
(iv) What you get if the functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ is fully faithful?

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Exercise 12. Consider the forgetful functor $f\colon \mathsf{Ab} \longrightarrow \mathsf{Grp}$. Show that the assignment $G \mapsto G/[G,G]$ induces a functor $\mathsf{abel}\colon \mathsf{Grp} \longrightarrow \mathsf{Ab}$, called the abelianization functor, which is a left adjoint of f, that is, (abel, f) is an adjoint pair. Does the functor f has a right adjoint? Note that $[G,G]$ denotes the commutator subgroup of the group G.

Exercise 13. Let R be a ring and X a set. Consider the following:

$$R^{(X)} = \big\{\text{functions } f\colon X \longrightarrow R \mid f(x) \neq 0 \text{ for only finitely many } x \in X\big\}$$

Show that:

(i) $R^{(-)}\colon \mathsf{Set} \longrightarrow \mathsf{Mod}\text{-}R$ is a functor.
(ii) There is an adjoint pair $(R^{(-)}, f)$, where $f\colon \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Set}$ is the forgetful functor.

Exercise 14. Show that in the category Set of sets, the product is the cartesian product and the coproduct is the disjoint union.

Exercise 15. Let $\mathscr{C}$ be a category.

(i) Show that the pullback of a diagram in $\mathscr{C}$

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{\;f\;} & Z \end{array}$$

is unique up to isomorphism. Prove the same for pushout diagrams in $\mathscr{C}$.
(ii) If $\mathscr{C} = \mathsf{Set}$, show that the set $W = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$ is the pullback of the above diagram.
(iii) What is the pushout in the category Set of sets?

Exercise 16. Exercises I.1 – I.14 from the notes.

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