MA3203 - Exercise sheet 9

Throughout k denotes a field.

1. Show that $\operatorname{rad} k[x] = 0$.

   Hint: Let $\lambda \in k[x]$ be a nonzero element. Show that $1 - \lambda x$ is not invertible in $k[x]$. Conclude using Proposition 16 from the lecture.

2. [2, Exercise 8.2] Let Γ be a finite and acyclic quiver and let $\Lambda = k\Gamma$. The aim of this exercise is to show that $\operatorname{rad} \Lambda = R_\Gamma$ where R_Γ is the ideal generated by the set of arrows of Γ.

   (a) Let $\alpha \in \Gamma_1$, i.e. α is an arrow in Γ. Show that $\alpha \in \operatorname{rad} \Lambda$.
   Hint: Show that $(y\alpha)^2 = 0$ for any $y \in \Lambda$. Use this to show that $1 - y\alpha$ is left invertible for any $y \in \Lambda$. Deduce that $R_\Gamma \subseteq \operatorname{rad} k\Gamma$.

   (b) Show that Λ / R_Γ is a semisimple Λ-module.
   Hint: Show that the quotient is isomorphic to a product of one-dimensional Λ-modules, where the product is taken over the vertices of Γ.

   (c) Use the equality $\operatorname{rad} \Lambda = \bigcap_{S \text{ simple}} \operatorname{Ann}_\Lambda S$ and part (b) to deduce that $\operatorname{rad} k\Gamma \subseteq R_\Gamma$.

   (d) Combining (a) and (c) we get that $\operatorname{rad} k\Gamma = R_\Gamma$.

3. Is it true that $\operatorname{rad} k\Gamma = R_\Gamma$ for any finite quiver Γ?

   Hint: Use Exercise 1.

4. [1, Lemma 1.3] Let Λ be a ring. In this exercise we give some more equivalent characterizations of $\operatorname{rad} \Lambda$.

(a) Show that $\lambda \in \operatorname{rad}\Lambda$ if and only if for any $x \in \Lambda$ the element $1 - x\lambda$ has a *two-sided inverse*, i.e. an element $y \in \Lambda$ such that $y(1 - x\lambda) = 1 = (1 - x\lambda)y$.

Hint: we know that there exists an element $y \in \Lambda$ such that $y(1 - x\lambda) = 1$. Show that $y = 1 + yx\lambda$ and deduce that y has a left inverse z. Then show that $z = 1 - x\lambda$.

(b) Show that $\lambda \in \operatorname{rad}\Lambda$ if and only if $1 - \lambda x$ has a two-sided inverse for any $x \in \Lambda$.

Hint: Show that the following hold for elements in Λ

- *If $(1 - cd)x = 1$, then $(1 - dc)(1 + dxc) = 1$.*
- *If $y(1 - cd) = 1$, then $(1 + dyc)(1 - dc) = 1$.*

5. [1, Lemma 1.3] Let Λ be a ring. Recall that the *opposite ring* of Λ, denoted Λ^{op}, is defined to be the ring with the same underlying set and group structure as Λ, but where the multiplication in Λ^{op} is defined by the formula $a * b = ba$.

(a) Show that a subset of Λ is a right ideal of Λ if and only if it is a left ideal in Λ^{op}. Deduce that $\operatorname{rad}\Lambda^{\mathrm{op}}$ is equal to the intersection of all maximal right ideals in Λ.

(b) Show that the following are equivalent for $\lambda \in \Lambda$:

i. $\lambda \in \operatorname{rad}\Lambda^{\mathrm{op}}$

ii. For any $x \in \Lambda$ the element $1 - \lambda x$ has a right inverse

iii. For any $x \in \Lambda$ the element $1 - \lambda x$ has a two-sided inverse

Hint: Use Exercise 4 *(a).*

(c) Using part (b) of the Exercise 3, show that $\operatorname{rad}\Lambda^{\mathrm{op}} = \operatorname{rad}\Lambda$.

6. Let $f : \Lambda \to \Lambda'$ be a morphism between k-algebras, and assume all simple left Λ'-modules have dimension 1 over k. Show that $f(\operatorname{rad}\Lambda) \subseteq \operatorname{rad}\Lambda'$.

Hint: Show that a simple Λ'-module must be a simple Λ-module, where the module structure is inherited from the map f. Show that if $\lambda \in \Lambda$ annihilates all simple Λ-modules, then $f(\lambda)$ annihilates all simple Λ'-modules. Conclude using Proposition 16 *from the lecture.*

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.