MA3203 - Exercise sheet 8

Throughout Λ denotes a ring.

1. (a) Show that the collection of Noetherian Λ-modules is closed under extensions.

   Hint: Let $0 \to K \to M \xrightarrow{p} N \to 0$ *be a short exact sequence with* K *and* L *noetherian, and let* $M_1 \subseteq M_2 \subseteq \cdots$ *be an increasing sequence of submodules of* M*. Then consider the sequence of submodules*
   $$K \cap M_1 \subseteq K \cap M_2 \subseteq \cdots$$
   of K*, and*
   $$p(M_1) \subseteq p(M_2) \subseteq \cdots$$
   of N*.*

   (b) Deduce that a Λ-module of finite length must be Noetherian.

   Hint: Use the fact that the collection of modules of finite length is the smallest class containing the simple modules and which is closed under extensions.

2. Let M be a semisimple Λ-module. Show that the following are equivalent:

   (a) M is Artinian.

   (b) M is Noetherian.

   (c) M has finite length.

   Hint: Show that all three conditions are equivalent to M *being a finite sum of simple modules.*

3. Let M be a Λ-module

(a) Show that if M is Noetherian, then M is finitely generated.

(b) Let Λ be a left Artinian ring, and assume M is finitely generated. Show that M has finite length.

Hint: Use that Λ has finite length (why?) and that there exists an epimorphism $\Lambda^n \to M \to 0$ (why?)

(c) Deduce that if Λ is left Artinian, then the following are equivalent:

(i) M has finite length.

(ii) M is Noetherian.

(iii) M is finitely generated.

(It is also true that this is equivalent to M being Artinian, but this is a bit more difficult to show).

4. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of Λ-modules.

(a) Show that M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

Hint: Assume M_2 is Noetherian. To show that M_3 is Noetherian, let $N_1 \subseteq N_2 \subseteq \cdots$ be an increasing sequence of submodules of M_3, and consider the sequence

$$p^{-1}(N_1) \subseteq p^{-1}(N_2) \subseteq \cdots$$

of submodules of M_2, where $p\colon M_2 \to M_3$ denotes the given epimorphism.

(b) Show that M_2 is Artinian if and only if M_1 and M_3 are Artinian.

Hint: This is proved using similar ideas as in part a)