MA3203 - Exercise sheet 7

Throughout Λ denotes a ring.

1. Let S and S' be simple Λ-modules, and let $f\colon S \to S'$ be a non-zero morphism of Λ-modules. Show that f is an isomorphism.

   Hint: Show that f is a monomorphism since S is simple, and f is an epimorphism since S' is simple

2. A sequence of morphisms of Λ-modules of the following form

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$$

   is called *exact* if f_1 is a monomorphism, f_n is an epimorphism, and $\operatorname{Im} f_i = \ker f_{i+1}$ for all $1 \leq i \leq n-1$.

   (a) Show that $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$ is exact if and only if there exist short exact sequences

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{g_2} K_2 \to 0$$
$$0 \to K_2 \xrightarrow{h_2} M_3 \xrightarrow{g_3} K_3 \to 0$$
$$\vdots$$
$$0 \to K_{n-2} \xrightarrow{h_{n-2}} M_{n-1} \xrightarrow{g_{n-1}} K_{n-1} \to 0$$
$$0 \to K_{n-1} \xrightarrow{h_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$$

   such that $f_i = h_i \circ g_i$ for $2 \leq i \leq n-1$.
   Hint: Set $K_i := \operatorname{Im} f_i$

(b) Deduce that if $0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \to 0$ is exact and M_i has finite length for all i, then

$$\sum_{i=1}^{n+1} (-1)^{i-1} \ell(M_i) = 0.$$

Hint: Use that $\ell(M_i) = \ell(K_i) + \ell(K_{i+1})$

3. [1, Exercise 4.2] Let M be a Λ-module of finite length, and let K, L be submodules of M. Show that $\ell(K + L) = \ell(K) + \ell(L) - \ell(K \cap L)$. (This generalizes the formula $\dim(V+W) = \dim V + \dim W - \dim V \cap W$ for vector spaces.)

   Consider the canonical map $K \oplus L \to M$. Show that it has image $K+L$ and kernel $K \cap L$. Deduce that we have an exact sequence

$$0 \to K \cap L \to K \oplus L \to K + L \to 0$$

   Now compare the length of the modules in this exact sequence.

4. (a) Show that the collection of finitely generated Λ-modules is closed under extensions.

   Hint: Let $0 \to K \to L \to M \to 0$ be a short exact sequence of modules, and assume K and L are finitely generated. Let $k_1, \ldots, k_m$ be a generating set for K and $m_1, \ldots, m_n$ be a generating set for M. Since the map $L \to M$ is surjective, we can find elements $l_1, \ldots, l_n$ mapping to $m_1, \ldots, m_n$. Then show that $k_1, \ldots, k_m, l_1 \ldots, l_n$ is a generating set for L.

   (b) Deduce that a Λ-module of finite length must be a finitely generated Λ-module.

   Hint: Use the fact that the collection of modules of finite length is the smallest class containing the simple modules and which is closed under extensions.

References

[1] L.-P. Thibault, 2019 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2019v/lecture_plan`.