MA3203 - Exercise sheet 6

Throughout k denotes a field

1. [1, Exercise I.13](Challenge) Let $A = M_n(k)$ be the ring of $n \times n$-matrices over k, and let M be an indecomposable A-module. Show that $l(M) = 1$ and $\dim_k M = n$.

   Hint: Show that A is a semisimple ring. This can be done using Wedderburn-Artin or in the following way: First show that

   $$A \cong \bigoplus_{i=1}^{n} S_i$$

   as a left A-module, where S_i is the ith column vector of A. Then show that S_i is a simple A-modules for all i (note that S_i and S_j are isomorphic for all i and j). This shows the claim.

   Conclude that any indecomposable A-module is simple and has to be isomorphic to one of the S_i's. Since $l(S_i) = 1$ and S_i has dimension n, this proves the claim.

2. Let Γ be an arbitrary quiver. Recall that a two-sided ideal $I \subseteq k\Gamma$ is called *admissible* if there exists an integer $m \geq 2$ so that $R_\Gamma^m \subseteq I \subseteq R_\Gamma^2$, where R_Γ is the arrow ideal of $k\Gamma$. We fix an admissible ideal I of $k\Gamma$.

   (a) (Challenge) Let M be a $k\Gamma/I$-module. Show that M is simple if and only if $\dim_k M = 1$.

   Hint: Choose a path p of maximal length such that $p \cdot m \neq 0$ for some element $m \in M$ (why does such a path exist?). Choose a vertex $i \in Q_0$ such that $e_i \cdot m \neq 0$ (why does such a vertex exist?). Set $m_i := e_i \cdot m$ and show that the one-dimensional subspace

   $$\{a \cdot m_i \mid a \in k\}$$

of M must be a Λ-submodule. Conclude that it must be equal to M, so M has dimension 1.

(b) Deduce that for any finite-dimensional $k\Gamma/I$-module M we have $l(M) = \dim_k M$.

(c) Let $\Gamma = 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$. Construct a simple $k\Gamma$-module of dimension greater than 1.

Show that the representation $k \underset{1}{\overset{1}{\rightleftarrows}} k$ *is simple.*

Extra problems

3 [1, Exercise III.6] Let $\Gamma = 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$ be the Kronecker quiver. Define a representation $M^{(n)}$ of Γ by

$$M^{(n)} = k[x]/x^n \underset{x}{\overset{1}{\rightrightarrows}} k[x]/x^n \ .$$

Show that $M^{(n)}$ is indecomposable.

This shows that $k\Gamma$ is not of finite representation type even when k is a finite field (compare with Problem 1 in Exercise sheet 3).

Hint: use that $M^{(n)}$ is indecomposable if and only if 0 *and* 1 *are the only idempotent endomorphism of $M^{(n)}$, see Problem 4 in Exercise Sheet 3..*

4 [2, Problem 5.2] Let Γ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Consider the representations of Γ

$$M : k^2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix}} k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2, \qquad N : k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}} k^2$$

(a) Find a composition series for each of M and N.

(b) What do you notice about the number of times each composition factor appears?

(c) (Challenge) How many different composition series does each of M and N have.

Hint: start by looking for subrepresentations of one smaller dimension.

5 [2, Problem 7.2] Let A_∞ be the "quiver" with vertex set $(A_\infty)_0 = \mathbb{Z}$ and an arrow $\alpha_i : i \to i+1$ for each $i \in \mathbb{Z}$. Define a representation (V, f) so that $V(i) = k$ and $f_{\alpha_i} = 1_k$ for all i.

(a) What are the subrepresentations of (V, f)?

Hint: The subrepresentations should be representations (W, g) *where* $W(i) = k \implies W(j) = k$ *for all* $i \leq j$

(b) Show that (V, f) does not have finite length.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.