

MA3203 - Exercise sheet 5

Throughout k denotes a field.

1. [2, Problem 5.1] Find a composition series for the module Λe_1 , where Λ is the path algebra of each of the following quivers (with relations):

(a) $\Lambda = k\Gamma$ for $\Gamma = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

(b) $\Lambda = k\Gamma$ for $\Gamma = 1 \xrightarrow{\alpha} 2 \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{array} 3$.

(c) $\Lambda = k\Gamma/(\beta\alpha)$ for $\Gamma = 1 \xrightarrow{\alpha} 2 \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{array} 3$.

(d) $\Lambda = k\Gamma/(\beta\alpha, \delta^3)$ for $\Gamma = 1 \xrightarrow{\alpha} 2 \begin{array}{c} \xrightarrow{\beta} \\ \xrightarrow{\gamma} \end{array} 3 \begin{array}{c} \circlearrowleft \\ \delta \end{array}$.

2. [1, Exercise I.5)] Let Λ be a ring, and let

$$0 \rightarrow L \xrightarrow{u} M \xrightarrow{r} N \rightarrow 0$$

be a short exact sequence of Λ -modules. Prove that u admits a retraction (a morphism $v: M \rightarrow L$ such that $v \circ u = 1_L$) if and only if r admits a section (a morphism $s: N \rightarrow M$ such that $r \circ s = 1_N$).

Hint: If v is a retraction of u , consider the morphism

$$t := 1_M - u \circ v: M \rightarrow M.$$

Show that t is 0 on L , and hence induces a morphism $s: N \rightarrow M$. Finally, show that $r \circ s = 1_N$. The other direction is proved similarly.

3. Here we consider modules of finite and infinite length.

- (a) Let Λ be a k -algebra. Show that if M is a Λ -module which has finite dimension as a k -vector space, then M has finite length.
- (b) Let $k[x]$ be the polynomial algebra in one variable. Show that $k[x]$ does not have finite length as a module over itself.
- (c) Given an example of a field k and an infinite-dimensional k -algebra which has finite length as a module over itself.

Hint: Let $k := \mathbb{Q}$ and consider \mathbb{R} as a \mathbb{Q} -algebra.

References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).
- [2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.