MA3203 - Exercise sheet 4

Throughout k denotes a field.

1. [4, Problem 2.1] Find the representations corresponding to the modules Λe_i for the different possible values of i and for the different cases of Λ listed below. Also, find the representation corresponding to Λ (as a left Λ-module) in each case:

   Hint: To compute Λ as a left Λ-module, use that $\Lambda = \Lambda e_1 \oplus \Lambda e_2 \oplus \Lambda e_3$ in (a) and (b), and that $\Lambda = \Lambda e_1 \oplus \Lambda e_2$ in (c).

   (a) $\Lambda = k\Gamma/(\rho)$ where Γ is the quiver

   $$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

   and $\rho = \{\beta\alpha\}$.

   (b) $\Lambda = k\Gamma/(\rho)$ where Γ is the quiver

   $$1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$$

   and $\rho = \{\beta\alpha\}$.

   (c) $\Lambda = k\Gamma/(\rho)$ where Γ is the quiver

   $$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \circlearrowleft \gamma$$

   and $\rho = \{\gamma\alpha, \gamma^3\}$.

2. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be an arbitrary quiver. The two-sided ideal of $k\Gamma$ generated by Γ_1 is called the *arrow ideal* of $k\Gamma$, and is denoted R_Γ. An arbitrary two-sided ideal $I \subseteq k\Gamma$ is called *admissible* if there exists an integer $m \geq 2$ so that $R_\Gamma^m \subseteq I \subseteq R_\Gamma^2$.

(a) Show that if I is admissible, then $k\Gamma/I$ is finite-dimensional (even if $k\Gamma$ is not).
Hint: Use that if $R_\Gamma^m \subseteq I$, then the dimension of $k\Gamma/I$ is finite if the dimension of $k\Gamma/R_\Gamma^m$ is finite.

(b) [1, Exercise II.7a] Let $\Gamma = {}_{\alpha}\circlearrowright 1 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 2$ and let $\rho = \{\alpha^2, \gamma\beta, \beta\gamma - \beta\alpha\gamma\}$. Show that (ρ) is an admissible ideal, and compute $\dim_k k\Gamma/(\rho)$
Hint: Show that $R_\Gamma^4 \subseteq (\rho)$.

(c) [1, Example 2.7] Let $\Gamma = {}_{\alpha}\circlearrowright 1 \circlearrowleft_{\beta}$ and let $\rho = \{\beta\alpha, \beta^2\}$. Show that (ρ) is not admissible and that $k\Gamma/(\rho)$ is infinite-dimensional.
Hint: Show that $\alpha^n \notin (\rho)$ for any $n > 0$

3. ([2, Problem 4.1]) Let $\Gamma = {}_{\alpha}\circlearrowright 1 \circlearrowleft_{\beta}$ and let $\Lambda = k\Gamma/(\alpha^2, \beta^2, \beta\alpha)$.

(a) Show that $(\alpha^2, \beta^2, \beta\alpha)$ is admissible, and compute the dimension of Λ.

(b) Let $k\langle x, y\rangle$ be the ring of polynomials with coefficients in k and noncommuting variables x and y (that is, $xy \neq yx$). Show that $\Lambda \cong k\langle x, y\rangle/(x^2, y^2, yx)$.

(c) Let $A = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & d & a \end{bmatrix} : a, b, c, d \in k \right\}$. Show that A is a ring and that $\Lambda \cong A$.
Hint: Define a morphism of rings $\phi \colon k\langle x, y\rangle \to A$ by sending x to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and y to $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that ϕ is surjective and induces a morphism $\overline{\phi} \colon k\langle x, y\rangle/(x^2, y^2, yx) \to A$. Compute the dimension of A and $k\langle x, y\rangle/(x^2, y^2, yx)$, and deduce that $\overline{\phi}$ is an isomorphism. Finally, use part (b) to conclude with the result.

4. Let Γ be the quiver $1 \circlearrowleft_{\alpha}$ and let $\rho = \{\alpha^n\}$. Using a similar argument as in the lecture, find all indecomposable representations of $k\Gamma/(\rho)$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.

[3] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, N. J., 1956

[4] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.