MA3203 - Exercise sheet 3

1. [1, Exercise III.8] Let $\Gamma = \; 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \;$ be the Kronecker quiver. Define a representation H_λ of Γ by

$$H_\lambda = \; k \underset{\lambda}{\overset{1}{\rightrightarrows}} k \; .$$

for every $\lambda \in k$. Show that H_λ is indecomposable and that $H_\lambda \cong H_\mu$ if and only if $\lambda = \mu$. This shows that $k\Gamma$ is not of finite representation type when k is infinite.

2. [4, Exercise 2.1] Let $\Gamma = \; 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \;$ and let

$$V = k^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} k^2$$

. Write V as a direct sum of indecomposable representations of Γ.

Hint: Show that V is isomorphi to a direct sum $V_1 \oplus V_2 \oplus V_3 \oplus V_4$ where

$$V_1 = k \to 0 \leftarrow 0 \quad \textit{and} \quad V_2 = 0 \to k \leftarrow 0$$
$$V_2 = 0 \to 0 \leftarrow k \quad \textit{and} \quad V_4 = k \xrightarrow{1} k \xleftarrow{1} k$$

3. Let $\Gamma = \; 1 \longrightarrow 2 \;$ and let $V_1 \to V_2$ be a representation of Γ. Using Problem 2 b) in the previous exercise sheet, show that we can write $V_1 \to V_2$ as a direct sum

$$V_1 \to V_2 \cong (k \to 0)^{m_1} \oplus (0 \to k)^{m_2} \oplus (k \xrightarrow{1} k)^{m_3}$$

for *unique* integers m_1, m_2, m_3. The uniqueness of this decomposition is a special case of the *Krull-Remak-Schmidt theorem*, which we will learn later in the course.

Hint: Choose a basis of V_1 and V_2, so that we get isomorphisms $k^m \cong V_1$ and $k^n \cong V_2$. Then $V_1 \to V_2$ is isomorphic to a representation of the form $k^m \xrightarrow{A' \cdot -} k^m$ for some $n \times m$ matrix A'. From Problem 2 b) in the previous exercise sheet it follows that $V_1 \to V_2$ is isomorphic to a representation $k^m \xrightarrow{A \cdot -} k^m$ where $A = \begin{pmatrix} I_{m_3} & 0_{m_3,m_1} \\ 0_{m_2,m_3} & 0_{m_2,m_1} \end{pmatrix}$ for unique integers m_1, m_2, m_3. Use this to deduce the result.

4. [2, Exercise 3.4] Let Γ be an arbitrary quiver and let V be a representation of Γ. Show that the following are equivalent.

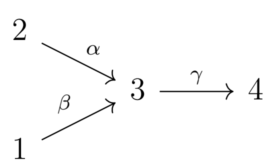
   (a) V is indecomposable.

   (b) If $\varphi : V \to V$ is idempotent ($\varphi^2 = \varphi$), then either $\varphi = 0$ (that is, $\varphi(i) = 0$ for all $i \in \Gamma_0$) or φ is an isomorphism[1].

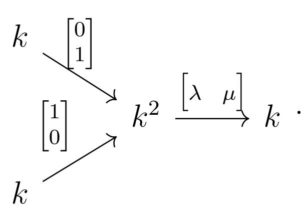
   Hint: If V is not indecomposable, then $V \cong V_1 \oplus V_2$ for two nonzero representations V_1 and V_2. Let ϕ be the map $V \xrightarrow{p} V_1 \xrightarrow{i} V$ where p is the projection and i is the inclusion.

   Conversely, if ϕ is an idempotent, then show that $V \cong \operatorname{Im} \phi \oplus \operatorname{Ker} \phi$.

5. [3, Exercise 2.5] Let Γ be the following quiver (of type D_4):



   . For $\lambda, \mu \in k$, let $M(\lambda, \mu)$ be the following representation of Γ



   (a) Determine when $M(\lambda, \mu)$ and $M(\lambda', \mu')$ are isomorphic.

[1]This property is sometimes stated as "End(V) contains no nontrivial idempotents".

(b) Determine for which values of λ and μ that $M(\lambda, \mu)$ is indecomposable.

Hint: Use Exercise 4.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.

[3] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.

[4] L.-P. Thibault, 2019 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2019v/lecture_plan`.