MA3203 - Exercise sheet 2

1. [3, Exercise 1.1] Let Γ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

   (a) Consider the representations of Γ

   $$(V,f) : k^2 \xrightarrow{\begin{bmatrix}1 & 1\end{bmatrix}} k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2, \qquad (V',f') : k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}} k^2$$

   and describe all morphisms $(V,f) \to (V',f')$ and $(V',f') \to (V,f)$.

   (b) Given two representations (W,g) and (W',g') of an arbitrary quiver Q, we say that (W',g') is a *subrepresentation* of (W,g) if $W'(i)$ is a subspace of $W(i)$ for every vertex $i \in Q_0$ and $g'_\alpha = g_\alpha|_{W'(i)}$ for every arrow $(\alpha : i \to j) \in Q_1$. Find all subrepresentations of the representation $k \xrightarrow{1} k \xrightarrow{1} k$ of the quiver Γ from part (a).

   (c) (Challenge) Give a definition of a *factor representation* and find all factor representations of the representation $k \xrightarrow{1} k \xrightarrow{1} k$ of the quiver Γ from part (a).

   Hint: A subrepresentation (W',g') of (W,g) could equivalently be defined as a morphism $(W',g') \to (W,g)$ of representations such that the map $W'(i) \to W(i)$ is a monomorphism for all $i \in Q_0$. The defintion of a factor representation should be the dual of this (i.e. reverse the arrow and replace "monomorphisms" by "epimorphisms").

2. In this exercise we investigate the connection between representations of quivers and certain matrix problems you might have seen in previous courses in linear algebra.

   (a) Recall that two $n \times m$ matrices A and B are *equivalent* if there exist an invertible $n \times n$ matrix P and an invertible $m \times m$ matrix

Q such that $B = PAQ^{-1}$. Let Γ be the quiver $1 \longrightarrow 2$. Show that A and B are equivalent if and only if the representations

$$k^m \xrightarrow{A\cdot -} k^n \quad \text{and} \quad k^m \xrightarrow{B\cdot -} k^n$$

of Γ are isomorphic.

Hint: use that $B = PAQ^{-1}$ if and only if we have a commutative square

$$\begin{array}{ccc} k^m & \xrightarrow{A\cdot -} & k^n \\ \downarrow{\scriptstyle Q\cdot -} & & \downarrow{\scriptstyle P\cdot -} \\ k^m & \xrightarrow{B\cdot -} & k^n \end{array}$$

(b) Let I_r denote the $r \times r$ identity matrix and let $0_{i,j}$ denote the $i \times j$ matrix with all entries 0. Show that any finite-dimensional representation of $1 \longrightarrow 2$ is isomorphic to a representation of the form $k^m \xrightarrow{A\cdot -} k^n$ where $A = \begin{pmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{pmatrix}$ for unique integers m, n, r.

Hint: Use (a) and what you know about elementary row and column operations from linear algebra).

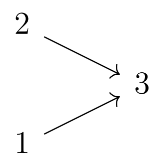
(c) Recall that two $n \times n$ matrices A and B are *similar* if there exists an invertible $n \times n$ matrix P such that $B = PAP^{-1}$. Let Γ be the quiver $1 \circlearrowright \alpha$. Show that A and B are similar if and only if the representations

$$k^n \circlearrowright A\cdot - \quad \text{and} \quad k^n \circlearrowright B\cdot -$$

of Γ are isomorphic.

(d) (Based on Section 1.2 in [1].) The *two subspace problem* is given by the set $\mathcal{M}$ of all pairs of matrices (A, B) with the same number of rows under the equivalence relation $(A, B) \sim (A', B')$ if and only if there exists invertible matrices P, Q, R such that $A' = PAQ^{-1}$ and $B' = PBR^{-1}$. Find a quiver Γ such that equivalence classes of $\mathcal{M}$ are in bijection with isomorphism classes of finite-dimensional representation of Γ.

Hint: Show that the quiver Γ *is*



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3. [2, Exercise 2.4] Let Γ be an arbitrary quiver and let $\phi\colon (V, f) \to (V', f')$ be a morphism of representations of Γ. We define the *kernel* of ϕ to be the representation $\ker(\phi) = (W, g)$ given by

$$W(j) = \ker(\phi_j) \qquad j \in \Gamma_0$$
$$g(\alpha) = f_\alpha|_{\ker(\phi_i)} \qquad (\alpha : i \to j) \in \Gamma_1$$

   (a) Show that $\ker(\phi)$ is well-defined; that is, that $\mathrm{Im}(f_\alpha|_{\ker(\phi_i)}) \subseteq \ker(\phi_j)$ for all arrows $\alpha : i \to j$.

   (b) Find the kernel of some of the morphisms you found in problem 1.

   (c) Give a definition of the *image* of a morphism of representations and find the image of some of the morphisms you found in problem 1.
   Hint: If $\phi\colon (V, f) \to (V', f')$ *is a morphism of representations, define the image pointwise, i.e* $(\mathrm{Im}\,\phi)(i) := \mathrm{Im}(\phi(i))$. *Using that* ϕ *is a morphism of representations, show that for each arrow* $\alpha\colon i \to j$ *the morphisms* $f'_\alpha\colon V'(i) \to V'(j)$ *restricts to a morphism* $f'_\alpha|_{\mathrm{Im}(\phi(i))}\colon \mathrm{Im}(\phi(i)) \to \mathrm{Im}(\phi(j))$. *Conclude that* $\mathrm{Im}\,\phi$ *is a subrepresentation of* (V', f').

   (d) Show that for any morphism of representations $\phi : (V, f) \to (V', f')$ there is a surjective morphism $(V, f) \twoheadrightarrow \mathrm{Im}\phi$ with kernel $\ker(\phi)$; that is, that $(V, f)/\ker\phi \cong \mathrm{Im}\phi$.

References

[1] M, Barot, *Introduction to the representation theory of algebras*, Springer, Cham (2015)

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.

[3] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2017v/ovinger.