MA3203 - Exercise sheet 19

Throughout k denotes a field.

1. [1, Exercise 5.2] Let $\Gamma = \; 1 \longrightarrow 2 \longrightarrow 3$ and let $\Lambda = k\Gamma$.

   (a) Find the indecomposable finitely-generated injective Λ-modules.
   Hint: Remember that the duality $D(-) = \mathrm{Hom}_k(-, k)$ gives a bijection between finitely generated indecomposable projective right Λ-modules and finitely generated indecomposable injective left Λ-modules. Now use that the finitely generated indecomposable projective right modules are $e_1\Lambda$ and $e_2\Lambda$ and $e_3\Lambda$ (why?). Apply $D(-)$.

   (b) Find the socles and injective envelopes of the following representations of Γ:

   i. $k \xrightarrow{0} k \xrightarrow{0} k$

   ii. $k \xrightarrow{1} k \xrightarrow{0} k$

   iii. $k^2 \xrightarrow{[1\ 0]} k \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} k^2$

   Hint: Use that the injective envelope of a representations is equal to the injective envelope of its socle.

2. [1, Exercise 5.3] Let $\Gamma = \; 1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 3$, let $\rho = \{\beta\alpha\}$, and let $\Lambda = k\Gamma/(\rho)$. Find the socles and injective envelopes of the following representations of (Γ, ρ):

   (a) $k \xrightarrow{1} k \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\overset{0}{\rightrightarrows}} k^2$

(b) $k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2 \underset{[1\ 1]}{\overset{[1\ 0]}{\rightrightarrows}} k$

(c) $0 \longrightarrow k^2 \underset{\begin{bmatrix}1&0\\1&0\end{bmatrix}}{\overset{\begin{bmatrix}1&1\\1&0\end{bmatrix}}{\rightrightarrows}} k^2$

Hint: Same strategy as in the previous exercise. It is a good idea to start by computing the finitely generated indecomposable injective Λ-modules.

3. (Exercise from the videos). Let Λ be an artin R-algebra, and let A, A_1, A_2 be finitely generated Λ-modules. Show the following

(a) $\operatorname{soc} A \cong \operatorname{Hom}_\Lambda(\Lambda/\mathsf{r}, A)$, where r is the radical of Λ.
Hint: Show that the homomorphism

$$\operatorname{Hom}_\Lambda(\Lambda/\mathsf{r}, A) \to A \quad f \mapsto f(1)$$

is injective with image $\operatorname{soc} A$.

(b) $\operatorname{soc}(A_1 \oplus A_2) \cong \operatorname{soc} A_1 \oplus \operatorname{soc} A_2$
Hint: Use a) and the fact that the Hom functor preserves finite direct sums.

Now assume Λ is a finite-dimensional algebra over k.

(c) Let X be in $\operatorname{mod}\Lambda$. Show that $\operatorname{soc} D(X) \cong D(X/\mathsf{r}X)$ in $\operatorname{mod}\Lambda^{\mathrm{op}}$.
Hint: There are two ways one can do this:

i. *First show that* $D(X/\mathsf{r}X) \cong \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda/\mathsf{r}, D(X))$ *in* $\operatorname{mod}\Lambda^{\mathrm{op}}$*, then use (a)*

ii. *or use that the socle of a module is its maximal semisimple submodule, and the top of a module (i.e. the quotient by the radical) is its maximal semisimple quotient.*

(d) Use (c) to show the following statement: If I is a finitely generated injective Λ-module, then I is indecomposable if and only if $\operatorname{soc} I$ is simple
Hint: Use that a finitely generated projective module P is indecomposable if and only if $P/\mathsf{r}P$ is simple.

4. (Here we give an alternative proof of Proposition 57 in the videos).

   (a) Let Λ be a ring, and let $i\colon X \to A$ be a monomorphism of Λ-modules. Show that the following are equivalent:

      (i) i is an essential monomorphism.

      (ii) For all morphisms $f\colon A \to B$ of Λ-modules, we have that f is a monomorphism if $f \circ i$ is a monomorphism

   Now assume Λ is a finite-dimensional algebra.

   (b) Let $p\colon A \to B$ be a morphism in $\operatorname{mod}\Lambda$. Show that p is an essential epimorphism in $\operatorname{mod}\Lambda$ if and only if $D(p)\colon D(B) \to D(A)$ is an essential monomorphism in $\operatorname{mod}\Lambda^{\mathrm{op}}$.

      Hint: Use (a).

   (c) Let $f\colon P \to A$ be a morphism in $\operatorname{mod}\Lambda$. Show that f is a projective cover in $\operatorname{mod}\Lambda$ if and only if $D(f)\colon D(A) \to D(P)$ is an injective envelope in $\operatorname{mod}\Lambda^{\mathrm{op}}$.

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.