MA3203 - Exercise sheet 18

Throughout k denotes a field.

1. (Exercise 13.2 in the videos. It is the same as problem 8 d) on exercise sheet 15, so you can skip it if you have done that problem). Let $\mathrm{vec}(k)$ be the category of finite-dimenional vector spaces over k. Given a finite-dimensional k-vector space V, consider $\varphi_V : V \to DD(V)$ so that for $x \in V$ and $f \in D(V)$, we have $\varphi_V(x)(f) = f(x)$.

   (a) Show that φ_V is an isomorphism.

   Hint: Show that φ_V is injective, and use that V and $DD(V)$ have the same dimension (why?)

   (b) Show that $\varphi = (\varphi_V)_{V \in \mathrm{vec}(k)}$ gives an isomorphism of functors $\mathrm{Id}_{\mathrm{vec}(k)} \to DD(-)$.

   Hint: From (a) you know that φ_V is an isomorphism, so you only need to show that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow{\scriptstyle \varphi_V} & & \downarrow{\scriptstyle \varphi_W} \\ DD(V) & \xrightarrow{DD(f)} & DD(W) \end{array}$$

   commutes for all linear transformations $f\colon V \to W$ between finite-dimensional k-vector spaces V and W. In other words, you need to show that $DD(f)(\varphi_V(v)) = \varphi_W(f(v))$. As a first step, you should understand how the morphism $DD(f)$ is defined.

2. (Exercise 13.3 in the videos) Let $f : V \to W$ be a morphism of finite-dimensional k-vector spaces. Let B and B' be bases for V and W, respectively, and let B^* and $(B')^*$ be the dual basis of $D(V)$ and $D(W)$,

respectively. Suppose the matrix form of f is $m_B^{B'}(f) =: A$. Show that $D(f) : D(W) \to D(V)$ has matrix form $m_{(B')^*}^{B^*} = A^\top$, where $A^\top$ denotes the transpose of A.

3. (Lemma 53 in the videos) Let Λ be a finite-dimensional k-algebra.

   (a) Let $f : A \to B$ and $g : B \to C$ be morphisms of finitely generated left Λ-modules. Show that the sequence

   $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   is exact if and only if the sequence

   $$0 \to D(C) \xrightarrow{D(g)} D(B) \xrightarrow{D(f)} D(A) \to 0$$

   is exact.

   (b) Let S be a finitely generated left Λ-module. Show that S is simple if and only if $D(S)$ is simple (as a module over Λ^{op}).

   Hint: Use a) and the fact that S is simple if and only if it is has no nontrivial submodules if and only if it is has nontrivial factors.

   (c) Let A be a finitely generated left Λ-module. Show that $\ell(A) = \ell(D(A))$.

   Hint: use a) and induction on the length of A. If the length is 1, then the result follows from b).

4. (Challenge) [1, Exercise III.3] Let $(\Gamma, \{\rho\})$ be a quiver with relations so that (ρ) is an admissible ideal of $k\Gamma$. Let Γ^{op} be the opposite quiver of Γ. This quiver has the same vertex set as Γ and for each $i \xrightarrow{\alpha} j$ an arrow of Γ there is an arrow $i \xleftarrow{\alpha^*} j$ of Γ^{op}. Now let $\{\rho^{op}\}$ be so that a linear combination of paths $\sum_i c^i \alpha_1^i \cdots \alpha_{m_i}^i \in \{\rho\}$ if and only if $\sum_i c^i (\alpha_{m_i}^i)^* \cdots (\alpha_1^i)^* \in \{\rho^{op}\}$.

   Now consider the equivalences of categories $G : \mathrm{Rep}(\Gamma, \{\rho\}) \to \mathrm{mod} k\Gamma/(\rho)$ and $F : \mathrm{mod} k\Gamma^{op}/(\rho^{op}) \to \mathrm{Rep}(\Gamma^{op}, \{\rho^{op}\})$. This gives a duality

   $$F \circ D \circ G : \mathrm{Rep}(\Gamma, \{\rho\}) \to \mathrm{Rep}(\Gamma^{op}, \{\rho^{op}\}).$$

   (a) Let (V, f) be a representation of $(\Gamma, \{\rho\})$. Show that $F \circ D \circ G(V, f) = (DV, Df)$, where for each vertex i in Γ^{op}, $DV_i := D(V_i)$ and for each arrow α^* in Γ^{op}, $Df_{\alpha^*} := D(f_\alpha)$.

(b) Let $\varphi : (V, f) \to (W, g)$ be a morphism in $\mathrm{Rep}(\Gamma, \{\rho\})$. Describe the morphism $F \circ D \circ G(\varphi)$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).