MA3203 - Exercise sheet 17

Throughout k denote a field

1. [2, Problem 4.1 and 4.2] Let Σ be a k-algebra.

   (a) Let Λ be a k-algebra and let $\mathcal{B}$ be a basis for Λ. Supposed $\varphi\colon \Lambda \to \Sigma$ is a morphism of k-vector spaces satisfying the following

   - $\varphi(1_\Lambda) = 1_\Sigma$.
   - $\varphi(b_1 b_2) = \varphi(b_1)\varphi(b_2)$ for all $b_1, b_2 \in \mathcal{B}$.

   Show that φ is a morphism of k-algebras

   Hint: We already know it is k-linear and sends the unit to the unit. Hence, we only need to show that it preserves the multiplicative structure, i.e. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \Lambda$. To show this, write x and y as a linear combination of the basis elements in $\mathcal{B}$, and use that φ is k-linear and satisfies $\varphi(b_1 b_2) = \varphi(b_1)\varphi(b_2)$ for all $b_1, b_2 \in \mathcal{B}$.

   (b) Let Γ be a quiver and let $f\colon \Gamma_0 \cup \Gamma_1 \to \Sigma$ be a function satisfying the following:

   - $\sum_{v \in \Gamma_0} f(v) = 1_\Sigma$.
   - $qr = 0$ (concatenation of paths) implies that $f(q)f(r) = 0$ for all $q, r \in \Gamma_0 \cup \Gamma_1$.
   - $f(v) = f(v)^2$ for all $v \in \Gamma_0$.
   - $f(e(\alpha))f(\alpha) = f(\alpha) = f(\alpha)f(s(\alpha))$ for all $\alpha \in \Gamma_1$, where $e(\alpha)$ and $s(\alpha)$ denotes the end and start of α, respectively

   Then f extends uniquely to a morphism of k-algebras $\tilde{f}\colon k\Gamma \to \Sigma$.

   Hint: Recall that $k\Gamma$ has a basis $\mathcal{B}$ (over k) given by the paths of Γ of length ≥ 0. Define a k-linear map $\varphi\colon k\Gamma \to \Sigma$ by defining it on $\mathcal{B}$ by

- *$\varphi(e_i) = f(i)$ for all vertices i.*
- *$\varphi(\alpha) = f(\alpha)$ for all arrows α.*
- *$\varphi(p) := f(\alpha_n)f(\alpha_{n-1})\cdots f(\alpha_1)$ for a path $p = \alpha_n\alpha_{n-1}\cdots\alpha_1$.*

Show that φ satisfies the conditions of part (a). Why is it unique?

2. (Used in the end of the proof of Theorem 50 in the lecture)

   Let Λ be a finite-dimensional algebra, and let r be the Jacobson radical of Λ. Let $x_1, \cdots x_n$ be elements of r, and assume their image $\overline{x_1}, \cdots \overline{x_n}$ in r/r^2 generate r/r^2 as a Λ/r-module. Show that $x_1, \cdots x_n$ generate r as a Λ-module

   Hint: Use Nakayama's lemma on r/N, where N is the submodule of r generated by $x_1, \cdots x_n$.

3. [1, Exercise II.16] This exercise shows that the assumption that k is algebraically closed is necessary in order to claim that every basic k-algebra is isomorphic to a quiver algebra.

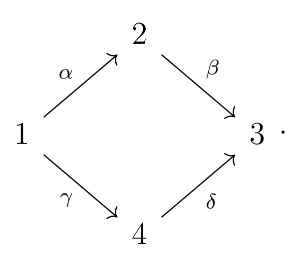
   (a) Show that $\mathbb{C}$ is a two-dimensional basic and connected $\mathbb{R}$-algebra.

   (b) Show that there is no quiver Q and admissible ideal I so that $\mathbb{C} \cong \mathbb{R}Q/I$.

   Hint: Show that if I is an admissible ideal of Q for which $\Gamma = \mathbb{R}Q/I$ is a 2-dimensional and connected $\mathbb{R}$-algebra, then $\Gamma = \mathbb{R}[x]/(x^2)$. Conclude by showing that $\mathbb{C}$ is not isomorphic to $\mathbb{R}[x]/(x^2)$.)

4. [1, Exercise II.17] Here we show that the generators of an admissible ideal are not uniquely determined in general:

   (a) Let Γ be the quiver



and let $\mathcal{I}_1 = \langle \beta\alpha + \delta\gamma \rangle$ and $\mathcal{I}_2 = \langle \beta\alpha - \delta\gamma \rangle$ be two ideals of $k\Gamma$. If $\mathsf{char} k \neq 2$, show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are admissible and distinct, and that there is an k-algebra isomorphism $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$.

Hint: Consider the map $f\colon \Gamma_0 \cup \Gamma_1 \to k\Gamma/\mathcal{I}_2$ *sending a vertex* v *to* e_v*, the arrow* α *to* $-\alpha$*, and the arrows* β*,* γ *and* δ *to themselves. Using Exercise* 1*, show that this gives a* k*-algebra homomorphism* $k\Gamma \to k\Gamma/\mathcal{I}_2$*. Furthermore, show that it vanishes of* $\mathcal{I}_1$ *and induces an isomorphism* $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$*.*

(b) Let Γ be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$ and let $\mathcal{I}_1 = \langle \gamma\alpha - \gamma\beta \rangle$ and $\mathcal{I}_2 = \langle \gamma\alpha \rangle$ be two ideals of $k\Gamma$. Show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are admissible and distinct, and that there is an k-algebra isomorphism $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$ (Here the characteristic of k can be arbitrary).

Hint: Consider the map $f\colon \Gamma_0 \cup \Gamma_1 \to k\Gamma/\mathcal{I}_2$ *sending a vertex* v *to* e_v*, the arrow* α *to* $\alpha + \beta$*, and the arrows* β *and* γ *to themselves. Using Exercise* 1*, show that this gives a* k*-algebra homomorphism* $k\Gamma \to k\Gamma/\mathcal{I}_2$*. Furthermore, show that it vanishes of* $\mathcal{I}_1$ *and induces an isomorphism* $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$*.*

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.