MA3203 - Exercise sheet 16

All the exercises are from [1, Problem set 15].

1. (From the introduction in the videos) Let Λ be a ring and let A be a left Λ-module. Let $\Gamma = \mathrm{End}_\Lambda(A)^{op}$.

   (a) For $a \in A$ and $f \in \mathrm{End}_\Lambda(A)^{op}$, define $a.f := f(a)$. Show that this makes A into a right Γ-module.

   (b) Show that $A = {}_\Lambda A_\Gamma$ is a Λ-Γ-bimodule; that is, $(\lambda.a).f = \lambda.(a.f)$ for all $\lambda \in \Lambda$, $a \in A$, and $f \in \Gamma$.

   (c) Let $X \in \mathrm{mod}\Lambda$. For $g \in \Gamma$ and $f \in \mathrm{Hom}_\Lambda(A, X)$, define $g.f \in \mathrm{Hom}_\Lambda(A, X)$ so that $(g.f)(a) := f(g(a))$ for all $a \in A$. Show that this makes $\mathrm{Hom}_\Lambda(A, X)$ into a left Γ-module.

2. (Lemma 47 in the videos) Let Λ be an arbitrary ring and let A_1, A_2, B_1, B_2 be left Λ-modules.

   (a) Show that there is an isomorphism of abelian groups

   $$\alpha : \mathrm{Hom}_\Lambda(A_1, B_1) \oplus \mathrm{Hom}_\Lambda(A_1, B_2) \to \mathrm{Hom}_\Lambda(A_1, B_1 \oplus B_2)$$

   given by $\alpha(f, g)(a) := (f(a), g(a))$ for $a \in A_1$.
   Hint: Its inverse is given by sending $f \in \mathrm{Hom}_\Lambda(A_1, B_1 \oplus B_2)$ to the composites $(p_1 \circ f, p_2 \circ f)$ where $p_1 \colon B_1 \oplus B_2 \to B_1$ and $p_2 \colon B_1 \oplus B_2 \to B_2$ are the canonical projections.

   (b) Show that there is an isomorphism of abelian groups

   $$\beta : \mathrm{Hom}_\Lambda(A_1, B_1) \oplus \mathrm{Hom}_\Lambda(A_2, B_1) \to \mathrm{Hom}_\Lambda(A_1 \oplus A_2, B_1)$$

   given by $\beta(f, g)(a_1, a_2) := f(a_1) + g(a_2)$ for $a_1 \in A_1$ and $a_2 \in A_2$.
   Hint: Its inverse is given by sending $f \in \mathrm{Hom}_\Lambda(A_1 \oplus A_2, B_1)$ to its restrictions $(f|_{A_1}, f|_{A_2})$ to A_1 and A_2, respectively.

(c) Let $\Gamma = \mathrm{End}_\Lambda(A_1)^{op}$. Show that the isomorphism α in part (a) is an isomorphism of left Γ-modules.

Hint: The action of Γ is defined in exercise 1 c). You need to show that the map commutes with this action.

3. (From the proof of Proposition 48 in the videos) Let Λ be a ring and let A, B, C, A', B', C' be Λ-modules.

   (a) Suppose there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{t_1} & B & \xrightarrow{t_2} & C \\ & & & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ 0 & \longrightarrow & A' & \xrightarrow{s_1} & B' & \xrightarrow{s_2} & C' \end{array}$$

   with both the top and bottom rows exact. Show that there exists a unique $f : A \to A'$ so that $g \circ t_1 = s_1 \circ f$. Moreover, show that if g and h are both isomorphisms, then so is f.

   (b) Suppose there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{t_1} & B & \xrightarrow{t_2} & C & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ A' & \xrightarrow{s_1} & B' & \xrightarrow{s_2} & C' & \longrightarrow & 0 \end{array}$$

   with both the top and bottom rows exact. Show that there exists a unique $h : C \to C'$ so that $h \circ t_2 = s_2 \circ g$. Moreover, show that if f and g are both isomorphisms, then so is h.

   Hint: In both problems you first need to show that $s_2 g t_1(a) = 0$ for any $a \in A$

4. (From the proof of Lemma 49 in the videos) Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Let X and Y be objects in $\mathcal{C}$. Show that $X \cong Y$ if and only if $F(X) \cong F(Y)$.

References

[1] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.