MA3203 - Exercise sheet 15

Most of Problem 1-6 and 9 is taken from Problem set 14 in [1]. Throughout k denotes a field.

1. *The categories of Rings and Sets.*

   (a) Show that there is a well-defined category Set whose objects are sets, morphisms are the usual maps between sets, and composition is the usual composition.

   (b) Show that there is a well-defined category Ring whose objects are rings (with $1 \neq 0$), morphisms are ring homomorphisms (sending 1 to 1), and composition is the usual composition.

   (c) Show that there is a forgetful functor $F\colon \text{Ring} \to \text{Set}$ sending a ring to its underlying set, and a morphism of rings to a map of the underlying sets.

2. *Monomorphisms and epimorphisms.*

   Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$. A morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ is called a *monomorphism* if for any object $C \in \mathcal{C}$ and any pair of morphisms $g_1, g_2 \in \text{Hom}_{\mathcal{C}}(C, A)$ we have $f \circ g_1 = f \circ g_2$ if and only if $g_1 = g_2$. Likewise, f is called an *epimorphism* if for any object $C \in \mathcal{C}$ and any pair of morphisms $g_1, g_2 \in \text{Hom}_{\mathcal{C}}(B, C)$ we have $g_1 \circ f = g_2 \circ f$ if and only if $g_1 = g_2$.

   (a) Let $\mathcal{C}$ be arbitrary. Show that any isomorphism in $\mathcal{C}$ is both a monomorphism and an epimorphism.

   Hint: Assume $f\colon Y \to Z$ is an isomorphism. To show that it is a monomorphism, you need to check that if $g_1\colon X \to Y$ and $g_2\colon X \to Y$ are morphisms such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. To do this, compose with the inverse of f and use associativity of the composition law.

(b) Let f be a morphism in Set or in $\operatorname{Mod}\Lambda$ for a ring Λ. Show that

(i) f is a monomorphism if and only if it is injective.

Hint: let $f\colon X \to Y$ be a monomorphism, and let $x, y \in X$ such that $f(x) = f(y)$:

Set*: Consider the maps $\{*\} \xrightarrow{x} X$ and $\{*\} \xrightarrow{y} X$ sending the one-point set $\{*\}$ to x and y respectively, and use the defining properties of a monomorphism to conclude $x = y$.*

$\operatorname{Mod}\Lambda$*: Consider the Λ-homomorphisms $\Lambda \xrightarrow{-\cdot x} X$ and $\Lambda \xrightarrow{-\cdot y} X$ sending the identity of Λ to x and y, respectively (how are they defined on an arbitrary element of Λ?). Use the defining properties of a monomorphism to conclude $x = y$.*

(ii) f is an epimorphism if and only if it is surjective.

Hint: Let $f\colon X \to Y$ be an epimorphism:

Set*: Let $y \in Y$. Consider the maps $Y \xrightarrow{g_1} \{*, y'\}$ and $Y \xrightarrow{g_2} \{*, y'\}$ where g_1 sends all elements to $*$, and where g_2 sends y to y' and all other elements to $*$. Use the defining properties of epimorphisms to conclude that $g_1 \circ f \neq g_2 \circ f$, and hence that y is in the image of f.*

$\operatorname{Mod}\Lambda$*: Let C be the quotient by the image of f, and let $g\colon Y \to C$ be the projection. Note that $g \circ f = 0 = 0 \circ f$. Use the defining property of an epimorphism to conclude that f is surjective.*

(iii) f is an isomorphism if and only if it is both a monomorphism and an epimorphism.

(c) Show that the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring, but not an isomorphism.

To show it is an epimorphism, use that for a ring morphism $g\colon \mathbb{Q} \to R$ we have $g(\frac{m}{n}) = g(m)g(n)^{-1}$ (why?).

(d) (Challenge) Show that every monomorphism in Ring is injective (but note that we have already shown that not every epimorphism in Ring is surjective!).

Hint: Let $\mathbb{Z}[x]$ be the ring of polynomials with integer coefficients. Show that a ring morphism $\mathbb{Z}[x] \to R$ is uniquely determined by choosing an element $r \in R$ (why?). Use this and a similar strategy as in Exercise 2 (b) (i) to prove the result.

(e) Show that $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$ in the category Set.

Hint: Use that both $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite (why?)

3. *Groups as categories*

   Let G be a group and let $\{*\}$ be a set with one element. Define a category $\mathcal{G}$ with a single object $*$, morphisms $\mathrm{Hom}_{\mathcal{G}}(*, *) = G$, and composition given by $g \circ h := gh$ for all $g, h \in \mathrm{Hom}_{\mathcal{G}}(*, *)$. Show that $\mathcal{G}$ is a well-defined category and that every $g \in \mathrm{Hom}_{\mathcal{G}}(*, *)$ is an isomorphism.

4. *Initial and terminal objects.*

   Let $\mathcal{C}$ be a category. An object I of $\mathcal{C}$ is called *initial* if for any object $A \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(I, A)$ contains a single element. Likewise, an object T of $\mathcal{C}$ is called *terminal* if for any object $A \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(A, T)$ contains a single element. If the same object is both initial and terminal, it is called a *zero object.*

   (i) Show that the trivial group is a zero object in the category Ab (and more generally in $\mathrm{Mod}\,\Lambda$ for any ring Λ).

   (ii) Show that $\mathbb{Z}$ is initial in Ring.

   (iii) (Challenge) Show that if A and B are both initial (resp. both terminal) in some category $\mathcal{C}$, then A and B are isomorphic.

   Hint: Consider the compositions $A \to B \to A$ and $B \to A \to B$, and use that there is a unique morphisms from an initial object to itself (resp. from a terminal object to itself).

5. *Posets*

   (a) Recall that a poset P is a set S together with a reflexive, antisymmetric, transitive relation $\leq$ (meaning that (1) for all $x, y, z \in P$, we have $x \leq x$ (2) if $x \leq y$ and $y \leq x$, then $x = y$, and (3) if $x \leq y$ and $y \leq z$ then $x \leq z$). An order-preserving map of posets $P \to Q$ is a map $f : P \to Q$ of sets so that if $x \leq y$ in P then $f(x) \leq f(y)$ in Q. Show that there is a well-defined category Pos whose objects are posets, morphisms are order-preserving maps, and composition is the usual composition.

   (b) Let P be a poset and let $\{*\}$ be a set with one element. Show that there is a well-defined category $\mathcal{P}$ whose objects are the elements of P and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{P}}(x,y) = \begin{cases} \{*\} & x \leq y \\ \emptyset & x \not\leq y. \end{cases}$$

What is the composition law in this category? What are the identity morphisms? Which morphisms are isomorphisms?

6. *The Hom-functor.*

Let $\mathcal{C}$ be any category and let A be an object in $\mathcal{C}$.

(a) Show that there is a covariant functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Set}$ which sends an object B of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and which sends a morphism $f : B \to C$ in $\mathcal{C}$ to the map $f_* : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$ given by $f_*(g) := f \circ g$.

(b) Show that there is a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,A) : \mathcal{C} \to \mathrm{Set}$ which sends an object B of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(B,A)$ and which sends a morphism $f : B \to C$ in $\mathcal{C}$ to the map $f^* : \mathrm{Hom}_{\mathcal{C}}(C,A) \to \mathrm{Hom}_{\mathcal{C}}(B,A)$ given by $f^*(g) := g \circ f$.

(c) Assume $\mathcal{C}$ is a R-category, where R is a commutative ring. Show that $\mathrm{Hom}_{\mathcal{C}}(A,-)$ as in (a) defines a covariant R-functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Mod}\,R$. Dually, show that $\mathrm{Hom}_{\mathcal{C}}(-,A)$ as in (b) defines a contravariant R-functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Mod}\,R$.

Hint: You need to show the following: (why?)

- $f_* : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$ *is a morphism of R-modules*
- *The association $f \mapsto f_*$ is a morphism of abelian groups*

$$\mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_R(\mathrm{Hom}_{\mathcal{C}}(A,B), \mathrm{Hom}_{\mathcal{C}}(A,C)).$$

7. *The functor category*

(a) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there exists a category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are morphisms of functors.

(b) Let Γ be a quiver, considered as a category. Show that $\mathrm{Fun}(\Gamma, \mathrm{Vec}(k))$ is equivalent to the category of representations of Γ in k.

8. *The opposite category*

   Let $\mathcal{C}$ be a category.

   (a) Show that there exists a category, denoted $\mathcal{C}^{\mathrm{op}}$, and given as follows

   - The objects of $\mathcal{C}^{\mathrm{op}}$ are the same as the objects of $\mathcal{C}$.
   - A morphism $f\colon C \to D$ in $\mathcal{C}^{\mathrm{op}}$ is the same as a morphism $f\colon D \to C$ in $\mathcal{C}$
   - A composite of morphism $g \circ f$ in $\mathcal{C}^{\mathrm{op}}$ is defined to be the composite $f \circ g$ in $\mathcal{C}$

   (b) Show that $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$.

   (c) Let $\mathcal{D}$ be a category. Show that a contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

   (d) Let $\mathrm{vec}(k)$ denote the category of finite-dimensional vector spaces. Show that the contravariant functor $\mathrm{Hom}_k(-,k)\colon\ \mathrm{vec}(k) \to \mathrm{vec}(k)$ gives an equivalence $\mathrm{vec}(k)^{\mathrm{op}} \xrightarrow{\cong} \mathrm{vec}(k)$.

   Hint: For each k-vector space V show that there is a morphism of k-vector spaces $\mathrm{ev}_V\colon V \to \mathrm{Hom}_k(\mathrm{Hom}_k(V,k),k)$ sending $v \in V$ to the morphism $\mathrm{ev}_V(v)\colon\ \mathrm{Hom}_k(V,k) \to k$ given by $\mathrm{ev}_V(v)(f) = f(v)$. Show that ev_V is an isomorphism if V is finite-dimensional. Finally, show that the collection of morphisms ev_V for V in $\mathrm{vec}(k)$ gives an isomorphism of functors $\mathrm{Id}_{\mathrm{vec}(k)} \to \mathrm{Hom}_k(-,k)\circ\mathrm{Hom}_k(-,k)$.

9. *Left exact functors*

   Let Λ be a ring and let $\mathrm{Mod}\,\Lambda$ be the category of left Λ-modules. A covariant functor $F : \mathrm{Mod}\,\Lambda \to \mathrm{Ab}$ is called *left exact* if for any short exact sequence

   $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   of Λ-modules the sequence

   $$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

   is exact (this means that $F(f)$ is injective and $\ker F(g) = \mathrm{Im}F(f)$). If in addition $F(g)$ is surjective, then F is called *exact*.

   Let M be a left Λ-module. Show that $\mathrm{Hom}_\Lambda(M,-) : \mathrm{Mod}\,\Lambda \to \mathrm{Ab}$ is always left exact and is exact if and only if M is projective.

Hint: For the last part, note that $\mathrm{Hom}_\Lambda(M,-)$ *is exact if and only if it sends epimorphisms to epimorphisms. Spell out what it means for* $\mathrm{Hom}_\Lambda(M,-)$ *to preserve epimorphisms, and show that it gives precisely the condition for* M *to be projective.*

10. *Morita equivalence*

    Let Λ_1 and Λ_2 be rings. We say that Λ_1 and Λ_2 are *Morita equivalent* if the module categories $\mathrm{Mod}\,\Lambda_1$ and $\mathrm{Mod}\,\Lambda_2$ are equivalent as $\mathbb{Z}$-categories.

    (a) Let $F\colon \mathrm{Mod}\,\Lambda_1 \to \mathrm{Mod}\,\Lambda_2$ be an equivalence of $\mathbb{Z}$-categories, and let M be a Λ_1-module. Show the following:

    (i) F preserves monomorphisms and epimorphisms.

    (ii) $M \neq 0$ if and only if $F(M) \neq 0$.

    (iii) M is a simple Λ_1-module if and only $F(M)$ is a simple Λ_2-module

    (iv) M is a projective Λ_1-module if and only $F(M)$ is an projective Λ_2-module

    (v) M is an indecomposable Λ_1-module if and only $F(M)$ is an indecomposable Λ_2-module
    Hint: First show that $F(M_1 \oplus M_2) \cong F(M_1) \oplus F(M_2)$ *in the following way: Let* $i_1\colon M_1 \to M_1 \oplus M_2$ *and* $i_2\colon M_2 \to M_1 \oplus M_2$ *be the inclusions and* $p_1\colon M_1 \oplus M_2 \to M_1$ *and* $p_2\colon M_1 \oplus M_2 \to M_2$ *be the projections. Show that* $F(i_1)\colon F(M_1) \to F(M_1 \oplus M_2)$ *and* $F(i_2)\colon F(M_2) \to F(M_1 \oplus M_2)$ *induces a morphism*

    $$g\colon F(M_1) \oplus F(M_2) \to F(M_1 \oplus M_2).$$

    Similarly, show that $F(p_1)\colon F(M_1 \oplus M_2) \to F(M_1)$ *and* $F(p_2)\colon F(M_1 \oplus M_2) \to F(M_2)$ *induces a morphism*

    $$h\colon F(M_1 \oplus M_2) \to F(M_1) \oplus F(M_2)$$

    Finally, show that g *and* h *are mutually inverse isomorphisms.*

    (b) Let $M_n(k)$ be the ring of $n \times n$ matrices over the field k, and let

    $$F\colon \mathrm{Mod}\,M_n(k) \to \mathrm{Mod}(k)$$

be the forgetful functor which sends a $M_n(k)$-module to its underlying vector space and a morphism of $M_n(k)$-module to the underlying morphism of vector spaces. Show that F is *not* an equivalence.

Hint: Show that the simple $M_n(k)$-module is sent to k^n, and hence the functor does not preserve simple objects. Conclude using the previous exercise.

(c) (Challenge) Show that $M_n(k)$ and k are Morita equivalent.

Hint: Let S be the unique simple $M_n(k)$-module, and consider the Hom-functor $\operatorname{Hom}_{M_n(k)}(S, -)\colon \operatorname{Mod} M_n(k) \to \operatorname{Mod} k$. Show that this functor is full, faithful, and dense. Here you need to use that since $M_n(k)$ is semisimple with S being the unique simple module, any $M_n(k)$-module is isomorphic to a direct sum of S with itself a number of times (possibly infinite).

References

[1] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.