MA3203 - Exercise sheet 14

Throughout k denotes a field.

1. (Proposition 39 b) and c) from the videos) Let Λ be a ring.

   (b) Let $e = e_1 + \cdots + e_m \in \Lambda$ be a sum of orthogonal idempotents. Show that $\Lambda e \cong \bigoplus_{i=1}^{m} \Lambda e_i$.

   Hint: If $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = \gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_n e_n$, *then multiply both sides with* e_i *to deduce that* $\lambda_i e_i = \gamma_i e_i$ *(why?)*

   (c) Let $e \in \Lambda$ be a nonzero idempotent. Show that Λe is indecomposable if and only if e is primitive.

   Hint: If $\Lambda e = P_1 \oplus P_2$, *then* $e = e_1 + e_2$ *where* $e_1 \in P_1$ *and* $e_2 \in P_2$. *Show that* e_1 *and* e_2 *are orthogonal idempotents.*

2. [1, Problem 13.3 (d)-(g)] Let Λ be a ring. An idempotent $e \in \Lambda$ is called a *central* if $e\rho = \rho e$ for all $\rho \in \Lambda$.

   (a) Show that if e is a central idempotent, then $1 - e$ is also a central idempotent.

   (b) Show that if e is a central idempotent, then $e\Lambda$ and $(1-e)\Lambda$ are both left and right Λ-modules.

   (c) Show that if e is a central idempotent, then $e\Lambda$ and $(1-e)\Lambda$ are rings and $\Lambda \cong e\Lambda \times (1-e)\Lambda$ (as rings).

   Hint: Show that $e\Lambda$ *is closed under addition and multiplication, and that* e *acts as an identity element on it.*

   (d) We say Λ is *connected* if and only if there do not exist nonzero rings Λ' and Λ'' so that $\Lambda \cong \Lambda' \times \Lambda''$. Show that Λ is connected if and only if 0 and 1_Λ are its only central idempotents.

   Hint: If $\Lambda = \Lambda' \times \Lambda''$, *then* $1_\Lambda = e + (1_\Lambda - e)$ *where* $e \in \Lambda'$ *and* $(1_\Lambda - e) \in \Lambda''$. *Show that* e *is a central idempotents.*

3. *(Radical of* $\operatorname{mod}\Lambda$). Let Λ be a finite-dimensional k-algebra and let M be a finite-dimensional Λ-module. Recall that the endomorphism ring

$$\operatorname{End}_\Lambda(M) := \operatorname{Hom}_\Lambda(M, M)$$

is a finite-dimensional k-algebra with multiplication given by composition. For each pair of finite-dimensional left Λ-modules M and N we define

$$\operatorname{rad}_\Lambda(M,N) = \{g \in \operatorname{Hom}_\Lambda(M,N) \mid h \circ g \in \operatorname{rad}(\operatorname{End}_\Lambda(M)) \ \forall\ h \in \operatorname{Hom}_\Lambda(N,M)\}$$

where $\operatorname{rad}(\operatorname{End}_\Lambda(M))$ denotes the radical of the ring $\operatorname{End}_\Lambda(M)$. We call $\operatorname{rad}_\Lambda$ the *(Jacobson) radical* of $\operatorname{mod}\Lambda$.

(a) Show that $\operatorname{rad}_\Lambda(M,N)$ is a sub-vector space of $\operatorname{Hom}_\Lambda(M,N)$

(b) Show that $\operatorname{rad}_\Lambda(M,M) = \operatorname{rad}(\operatorname{End}_\Lambda(M))$.

(c) Show that $\operatorname{rad}_\Lambda(M,N) = \operatorname{Hom}_\Lambda(M,N)$ if M and N are indecomposable and $M \ncong N$.
Hint: If $\operatorname{rad}_\Lambda(M,N) \neq \operatorname{Hom}_\Lambda(M,N)$*, then there exists* $g\colon M \to N$ *and* $h\colon N \to M$ *such that* $h \circ g \notin \operatorname{rad}(\operatorname{End}_\Lambda(M))$*. Show that there exists* $h' \in \operatorname{End}_\Lambda(M)$ *such that* $h' \circ h \circ g = 1_M$*. Conclude that* $e = g \circ h' \circ h\colon N \to N$ *is an idempotent, and therefore an isomorphism (why?). Conclude that* g *must be an isomorphism between* M *and* N *(why?).*

(d) Let $f \in \operatorname{Hom}_\Lambda(N,L)$ and $g \in \operatorname{rad}_\Lambda(M,N)$. Show that $f \circ g \in \operatorname{rad}_\Lambda(M,L)$.

(e) Let $f \in \operatorname{Hom}_\Lambda(K,M)$ and $g \in \operatorname{rad}_\Lambda(M,N)$. Show that $g \circ f \in \operatorname{rad}_\Lambda(K,N)$.
Hint: First show that $h \circ g \circ f \in \operatorname{rad}(\operatorname{End}_\Lambda(K))$ *for all* $h \in \operatorname{Hom}_\Lambda(N,K)$ *if and only if* $h \circ g \circ f$ *is a nilpotent endomorphism for all* $h \in \operatorname{Hom}_\Lambda(N,K)$*. Then use that* $f \circ h \circ g$ *is nilpotent since* $f \circ h \circ g \in \operatorname{rad}(\operatorname{End}_\Lambda(M))$

(f) Show that

$$\operatorname{rad}(M, N_1 \oplus N_2) = \operatorname{rad}(M, N_1) \oplus \operatorname{rad}(M, N_2)$$

$$\operatorname{rad}(M_1 \oplus M_2, N) = \operatorname{rad}(M_1, N) \oplus \operatorname{rad}(M_2, N).$$

Hint: To show $\operatorname{rad}(M, N_1 \oplus N_2) \subseteq \operatorname{rad}(M, N_1) \oplus \operatorname{rad}(M, N_2)$*, use (d) and the projections* $p_1 \colon N_1 \oplus N_2 \to N_1$ *and* $p_2 \colon N_1 \oplus N_2 \to N_2$*. To show* $\operatorname{rad}(M, N_1) \oplus \operatorname{rad}(M, N_2) \subseteq \operatorname{rad}(M, N_1 \oplus N_2)$*, use (d) and the inclusions* $i_1 \colon N_1 \to N_1 \oplus N_2$ *and* $i_2 \colon N_2 \to N_1 \oplus N_2$*.*

The equality $\operatorname{rad}(M_1 \oplus M_2, N) = \operatorname{rad}(M_1, N) \oplus \operatorname{rad}(M_2, N)$ *is proved similarly, using (e) instead of (d).*

4. Let Λ be a finite-dimensional k-algebra and let M and N be finite-dimensional Λ-modules. We define $\operatorname{rad}^2_\Lambda(M, N)$ to be the sub-vector space of $\operatorname{Hom}_\Lambda(M, N)$ generated by all elements of the form $g \circ f$ where $f \in \operatorname{rad}_\Lambda(M, K)$ and $g \in \operatorname{rad}_\Lambda(K, N)$, where K is an arbitrary finite-dimensional Λ-module. The quotient

$$\operatorname{Irr}(M, N) = \operatorname{rad}_\Lambda(M, N) / \operatorname{rad}^2_\Lambda(M, N)$$

is called the *space of irreducible morphisms*. The *Auslander–Reiten quiver* Γ_Λ of Λ is defined as follows

- The vertices of Γ_Λ are the isomorphism classes $[M]$ of finite-dimensional indecomposable Λ-modules.
- The number of arrows $[M] \to [N]$ is equal to the dimension of $\operatorname{Irr}(M, N)$

One of the main goals in the representation theory of finite-dimensional algebras is to determine the Auslander–Reiten quiver Γ_Λ of a finite-dimensional algebra Λ, since it contains a lot of information about the module category of Λ.

(a) Determine the Auslander–Reiten quiver for the algebra $k\Gamma$ where $\Gamma = (1 \xrightarrow{\alpha} 2)$

Hint: Its indecomposable modules are $k \to 0$, $0 \to k$ *and* $k \xrightarrow{1} k$

(b) Determine the Auslander–Reiten quiver for the algebra $k[x]/(x^2)$.
Hint: Its indecomposable modules are k *(with* x *acting as* 0*) and* $k[x]/(x^2)$*.*

(c) Let $\Lambda = k\Gamma$ where $\Gamma = (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$. Assume that we know that

$$S_1 = (k \to 0 \to 0) \quad S_2 = (0 \to k \to 0) \quad S_3 = (0 \to 0 \to k)$$

$$\Lambda e_1 = (k \xrightarrow{1} k \xrightarrow{1} k) \quad \Lambda e_2 = (0 \xrightarrow{0} k \xrightarrow{1} k)$$

$$I = (k \xrightarrow{1} k \xrightarrow{0} 0)$$

are the indecomposable Λ-modules, up to isomorphism. Determine the Auslander–Reiten quiver of Λ.

References

[1] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.