MA3203 - Exercise sheet 13

Throughout k denotes a field

1. [2, Exercise 12.1] Let $\Gamma = \; 1 \rightrightarrows 2$. For $\lambda \in k$ define a representation $M(\lambda) = \; k \overset{1}{\underset{\lambda}{\rightrightarrows}} k$ and define $M(\infty) = \; k \overset{0}{\underset{1}{\rightrightarrows}} k$. For all $\lambda \in k \cup \{\infty\}$, show that there is a projective cover $\Lambda e_1 \xrightarrow{f_\lambda} M(\lambda)$ and find $\ker(f_\lambda)$.

2. Let Λ be a ring, and let e be an idempotent of Λ.

   (a) Let M be a Λ-module. Show that we have an isomorphism

   $$\mathrm{Hom}_\Lambda(\Lambda e, M) \cong eM$$

   of abelian groups.
   Hint: Show that the map

   $$\mathrm{Hom}_\Lambda(\Lambda e, M) \to eM \quad f \mapsto f(e)$$

   is well defined, Λ*-linear, and bijective.*

   (b) Show that $e\Lambda e$ inherits a ring structure from Λ with identity element e. Furthermore, show that with this structure

   $$\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)^{\mathrm{op}} \cong e\Lambda e$$

   becomes an isomorphism of rings, where $\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)^{\mathrm{op}}$ denotes the opposite ring of the endomorphism ring $\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)$. Also, show that

   $$\mathrm{Hom}_\Lambda(\Lambda e, M) \cong eM$$

   becomes an isomorphism of left $e\Lambda e$-modules.

Hint: The additive and multiplicative structure of $e\Lambda e$ is inherited from Λ. The isomorphism

$$\operatorname{Hom}_\Lambda(\Lambda e, \Lambda e)^{\mathrm{op}} \xrightarrow{\cong} e\Lambda e$$

is given by the map in a) with $M = \Lambda e$. The action of $e\Lambda e$ on $\operatorname{Hom}_\Lambda(\Lambda e, M)$ is given by $(x \cdot f)(y) = f(y \cdot x)$, and on eM just by left multiplication.

(c) [2, Exercise 12.4 (b)] Suppose $\Lambda = k\Gamma/I$ for some quiver Γ and admissible ideal I, and let $i \in \Gamma_0$ be a vertex. Show that $\dim_k \operatorname{Hom}_\Lambda(\Lambda e_i, M) = \dim_k M(i)$, where $M(i)$ is the vector space at vertex i when M is considered as a representation.
Hint: Use that $e_i M \cong M(i)$

3. [1, Exercise I.7.7] let $\Lambda = k[t]$. Show that the Λ-module $k[t]/(t^3)$ has no projective cover in $\operatorname{Mod}\Lambda$

   Hint: If $f\colon P \to k[t]/(t^3)$ is a projective cover, then P has to be a summand of all projective modules Q for which there exists an epimorphism $Q \to k[t]/(t^3)$ (why?). Deduce that $P \cong k[t]$.

   Now consider the morphism $g\colon k[t] \to k[t]$ given by $1 \mapsto 1 + t^3$. Show that $f \circ g$ is an epimorphism, but g itself is not an epimorphism. Conclude that this gives a contradiction.

4. [2, Exercise 12.3] Let Λ be a ring and M a module. A *projective resolution* of M is a sequence

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

   where each P_i is projective, $\ker(f_i) = \operatorname{Im}(f_{i+1})$ for all $i \geq 0$, and f_0 is surjective. Often we use the symbol $P_\bullet$ to denote a projective resolution.

   If there exists some m so that $P_m \neq 0$ and $P_n = 0$ for all $n > m$, then we say that the length of the projective resolution is m. The *projective dimension* of M is then defined to be the minimum possible length of a projective resolution for M.

   There is a theorem which says that if $P_\bullet$ is a projective resolution of M so that $P_0 \xrightarrow{f_0} M$ is a projective cover and $P_{i+1} \xrightarrow{f_{i+1}} \ker(f_i)$ is a projective cover for all $i \geq 0$, then the projective dimension of M is equal to the length of $P_\bullet$.

(a) For an arbitrary ring Λ and module M, show that the projective dimension of M is 0 if and only if M is projective.

(b) Let $\Gamma = \; 1 \longrightarrow 2$ and let $\Lambda = k\Gamma$. Find the projective dimensions of the modules $\Lambda e_1, \Lambda e_2$, and $S_1 = \Lambda e_1 / \Lambda e_2$.

Hint: Show that there exists an exact sequence

$$0 \to \Lambda e_2 \to \Lambda e_1 \to S_1 \to 0.$$

(c) Let $\Gamma = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and let $\Lambda = k\Gamma/(\alpha\beta, \beta\alpha)$. Find the projective dimensions of the simple modules S_1 and S_2.

Hint: Show that there exists exact sequences

$$\cdots \to \Lambda e_1 \to \Lambda e_2 \to \Lambda e_1 \to S_1 \to 0$$

and

$$\cdots \to \Lambda e_2 \to \Lambda e_1 \to \Lambda e_2 \to S_2 \to 0.$$

(d) Let Λ be a left artinian ring. The *left global dimension* of Λ is the supremum of the projective dimensions of all finitely generated left Λ-modules[1]. Show that if Λ is *left hereditary* (that is, every submodule of a finitely generated projective left module is projective) then the global dimension of Λ is at most 1.[2]

(e) (Challenge) Let Γ be a finite acyclic quiver. For an arrow $\alpha \in \Gamma_1$ let $s(\alpha)$ and $t(\alpha)$ denote its source and target of α, respectively. Show that a finite-dimensional representation (V, f) of Γ is projective if and only if the map

$$\bigoplus_{\alpha \in \Gamma_1,\, t(\alpha)=i} V_{s(\alpha)} \xrightarrow{(f_\alpha)} V_i$$

is a monomorphism for all $i \in \Gamma_0$. Deduce that $k\Gamma$ is left hereditary.

[1] It can be shown that this is the same as the supremum of all Λ-modules.

[2] The converse to this statement is also true.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.