MA3203 - Exercise sheet 12

1. [1, Exercise 4.1] Let $Q = \ 1 \longrightarrow 2 \longrightarrow 3$. Find the projective cover and the kernel of the projective cover of each of the following representations of Q.

   (a) $k \xrightarrow{0} k \xrightarrow{0} k$

   (b) $k \xrightarrow{1} k \xrightarrow{0} k$

   (c) $k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xrightarrow{[1\ 1]} k$

   Hint: There are several ways of solving this problem, here is one suggestion.

   First determine the projectives kQe_1, kQ_2 and kQe_3. Then for each representation M above try to construct an epimorphism $p\colon P \to M$ where $P = (kQe_1)^n_1 \oplus (kQe_2)^{n_2} \oplus (kQe_3)^{n_3}$. Then apply the top functor to p. If this becomes an isomorphism, you are done (why?). If not, see if you can remove summands from P while still retaining an epimorphism. Then apply the top again and repeat the process.

2. [1, Exercise 4.2] Let $Q = \ 1 \longrightarrow 2 \rightrightarrows 3$. Find the projective cover and the kernel of the projective cover of each of the following representations of Q.

   (a) $k \xrightarrow{1} k \overset{0}{\underset{\begin{bmatrix}0\\1\end{bmatrix}}{\rightrightarrows}} k^2$

   (b) $k \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} k^2 \overset{[1\ 0]}{\underset{[1\ 1]}{\rightrightarrows}} k$

(c) $$0 \longrightarrow k^2 \underset{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}}{\overset{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}}{\rightrightarrows}} k^2$$

Hint: See Problem 1

3. Let Λ be a left artinian ring, and let e be an idempotent in $\Lambda/\operatorname{rad}\Lambda$. Show that there exists an idempotent e' in Λ such that $\pi(e') = e$, where $\pi\colon \Lambda \to \Lambda/\operatorname{rad}\Lambda$ is the projection map.

   Hint: Use a similar argument as in the proof of Theorem 30 in the lecture

4. Let Λ be a left artinian ring. A morphism $f\colon M \to N$ of Λ-modules is called *right minimal* if any morphism $h\colon M \to M$ satisfying $f \circ h = f$ is an isomorphism.

   Let $f\colon P \to M$ be a surjective morphism of finitely generated Λ-modules, and assume P is projective. Show that f is an essential epimorphism if and only if f is right minimal.

   Hint: The fact that essential epimorphisms between finitely generated modules are right minimal follows from Problem 3 *on the previous exercise sheet. For the converse, assume you have a morphism $g\colon N \to P$ such that $f \circ g$ is an epimorphism. Then use that P is projective to get a morphism $h\colon P \to N$ satisfying $f \circ g \circ h = f$. Then use that f is right minimal to get the result (how?).*

References

[1] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.