MA3203 - Exercise sheet 10

Throughout k denotes a field.

1. [1, Proposition 3.7 (a)-(c)] Let Λ be a ring and let L, M, N be Λ-modules. Show that the following hold:

   (a) An element $m \in M$ belongs to $\operatorname{rad} M$ if and only if $f(m) = 0$ for any $f \in \operatorname{Hom}_\Lambda(M, S)$ and any simple Λ-module S.

   Hint: Use that if $f\colon M \to S$ is a nonzero morphism with S simple, then $\operatorname{Ker} f$ is a maximal submodule of M, and that if $N \subset M$ is a maximal submodule, then M/N is simple.

   (b) If $g\colon M \to N$ is a morphism of Λ-modules, then $g(\operatorname{rad} M) \subseteq \operatorname{rad} N$.

   Hint: Use a)

   (c) $\operatorname{rad}(M \oplus N) = \operatorname{rad}(M) \oplus \operatorname{rad}(N)$.

   Hint: Use b)

2. [1, Exercise I.7.6] Let Λ be a ring, let M be a Λ-module, and let N be a Λ-submodule of M. Prove that

   (a) $(N + \operatorname{rad} M)/N \subseteq \operatorname{rad}(M/N)$.

   Hint: Apply exercise 1b) to the morphism $M \to M/N$.

   (b) If $N \subseteq \operatorname{rad} M$, then $\operatorname{rad}(M/N) = (\operatorname{rad} M)/N$.

   Hint: by part a) we only need to show $\operatorname{rad}(M/N) \subseteq (\operatorname{rad} M)/N$. For this, use that the radical is the intersection of the maximal submodules, and that if $K \subseteq M$ is maximal, then $N \subseteq K$ and K/N is maximal in M/N (why?).

3. Find the radical of the module Λe_1, where Λ is the path algebra of each of the following quivers (with relations):

   Hint: Use that the radical of a representation is the subrepresentations generated by the image of the arrows, see video 5m

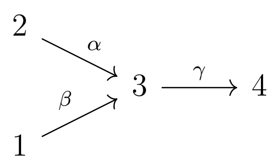
   (a) $\Lambda = k\Gamma$ for $\Gamma = \; 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$.

   (b) $\Lambda = k\Gamma/(\gamma\alpha - \gamma\beta)$ for $\Gamma = \; 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$.

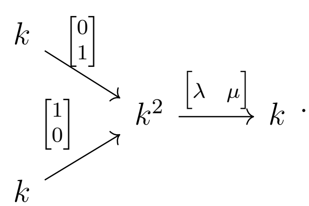
   (c) $\Lambda = k\Gamma$ for $\Gamma = \; 2 \xrightarrow{\alpha} 1 \xleftarrow{\beta} 3$.

   (d) $\Lambda = k\Gamma/(\alpha\beta, \beta^2)$ for $\Gamma = \; 2 \xleftarrow[\alpha]{} 1 \circlearrowright \beta$.

4. Let Γ be the following quiver



   and for $\lambda, \mu \in k$ let $M(\lambda, \mu)$ be the representation given by



   Compute the radical of $M(1, 1)$ and $M(0, 0)$.

   Hint: Use that the radical of a representation is the subrepresentations generated by the image of the arrows, see video 5m

5. [2, Exercise 9.2] Let Λ be a ring and let M be a module of finite length $\ell(M)$.

   (a) Show that $\operatorname{rad}^{\ell(M)} M = 0$.

(b) M is called *uniserial* if it has a unique composition series. Let m be the smallest positive integer so that $\operatorname{rad}^m M = 0$ (this is sometimes called the *radical length* of M). Show that M is uniserial if and only if its radical length is equal to its length.

(c) Find the radical length of the modules in exercise 3.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.