

# MA3203 - Exercise sheet 1

Throughout  $k$  denotes a field and  $\Gamma = (\Gamma_0, \Gamma_1)$  denotes a finite quiver.

1. (a) Let  $\Gamma$  be the quiver  $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ , and let  $R = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$  be a lower triangular matrix ring. Show that the map

$$\phi: k\Gamma \rightarrow R$$

given by

$$\begin{aligned} \phi(e_1) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \phi(e_2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \phi(e_3) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \phi(\alpha) &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \phi(\beta) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \phi(\beta\alpha) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}$$

is an isomorphism of algebras.

- (b) Let  $R$  denote the lower triangular matrix ring  $\begin{bmatrix} k & 0 & \cdots & 0 \\ k & k & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ k & k & \cdots & k \end{bmatrix}$  with  $n$ -rows and  $n$ -columns. Show that  $R$  is isomorphic to the path algebra of the quiver  $1 \rightarrow 2 \rightarrow \cdots \rightarrow n-1 \rightarrow n$ .

2. Let  $V$  be a finite-dimensional vector space over  $k$ .

- (a) Show that the set

$$\begin{bmatrix} k & 0 \\ V & k \end{bmatrix} := \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, c \in k \text{ and } b \in V \right\}$$

has a natural structure of a  $k$ -algebra.

*Hint: Show that the set  $\begin{bmatrix} k & 0 \\ V & k \end{bmatrix}$  is closed under addition and multiplication of matrices*

- (b) For  $V = k^2$ , show that  $R = \begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix}$  is isomorphic to the path algebra of the quiver  $\Gamma = 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$ .

*Hint: Let  $f_1$  and  $f_2$  be the basis vectors for  $k^2$ . Show that the map  $\phi: k\Gamma \rightarrow R$  given by*

$$\begin{aligned} \phi(e_1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \phi(e_2) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \phi(\alpha) &= \begin{pmatrix} 0 & 0 \\ f_1 & 0 \end{pmatrix} & \phi(\beta) &= \begin{pmatrix} 0 & 0 \\ f_2 & 0 \end{pmatrix} \end{aligned}$$

*is an isomorphism of algebras.*

- (c) For  $V$  arbitrary, can you find a quiver  $\Gamma$  such that  $\begin{bmatrix} k & 0 \\ V & k \end{bmatrix}$  is isomorphic to  $k\Gamma$ ?

*Hint: Show that  $\Gamma$  is the quiver with two vertices 1 and 2 and with  $n$  arrows from 1 to 2, where  $n$  is the dimension of  $V$ .*

We say that a quiver  $\Gamma$  is connected if for any two vertices there exists an *undirected* path between them. We say that an algebra  $A$  is *connected* if it cannot be written as product  $A_1 \times A_2$  of two algebras  $A_1$  and  $A_2$ .

- 3 In this exercise we prove Lemma 1.7 in [1], i.e. that  $\Gamma$  is connected if and only if  $k\Gamma$  is connected.

- (a) Let  $\Gamma$  and  $\Gamma'$  be two quivers, and let  $\Gamma \cup \Gamma'$  denote their disjoint union. Show that  $k\Gamma \times k\Gamma'$  and  $k(\Gamma \cup \Gamma')$  are isomorphic as  $k$ -algebras.
- (b) Deduce that if  $k\Gamma$  is connected as a  $k$ -algebra, then  $\Gamma$  is connected as a quiver
- (c) (Challenge) Assume  $k\Gamma = R_1 \times R_2$  for two  $k$ -algebras  $R_1$  and  $R_2$ . Let  $e_i$  be the trivial path at a vertex  $i \in \Gamma_0$ . Show that  $e_i \in R_1$  or  $e_i \in R_2$ .

*Hint: Do the following:*

- Write  $e_i = (f_1, f_2)$  where  $f_1 \in R_1$  and  $f_2 \in R_2$ . Show that  $f_1$  and  $f_2$  are idempotents satisfying  $f_1 f_2 = f_2 f_1 = 0$ .
- Write  $f_1 = \sum_{i=0}^n p_i$  and  $f_2 = \sum_{i=0}^n q_i$  where  $p_i$  and  $q_i$  are sums of paths of length  $i$  for each  $0 \leq i \leq n$ . Then show that either  $p_0 = e_i$  and  $q_0 = 0$ , or  $q_0 = e_i$  and  $p_0 = 0$ .
- Show that if an idempotent in  $k\Gamma$  is the sum of paths of length greater than or equal to 1, then it must be 0.

- (d) Show that if  $\Gamma$  is connected, then  $k\Gamma$  is connected. (Hint: use (c)).  
*Hint: Write  $k\Gamma = R_1 \times R_2$ . Using (c), show that any path  $p$  has to be either in  $R_1$  or  $R_2$ . Deduce that if there is a path between two vertices, then both vertices has to be in  $R_1$  or  $R_2$ . Deduce the claim.*

Recall that a ring  $R$  is *local* if it has a unique maximal left ideal.

4 [1, Exercise II.2cd] Let  $\Gamma$  be a connected quiver.

- (a) Show that  $k\Gamma$  is a local ring if and only if  $|\Gamma_0| = 1$  and  $|\Gamma_1| = 0$ .  
 (b) Show that  $k\Gamma$  is commutative if and only if  $|\Gamma_0| = 1$  and  $|\Gamma_1| \leq 1$ .

## References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).