MA3203 - Exercise sheet 9

Throughout k denotes a field.

1. Show that $\operatorname{rad} k[x] = 0$.

2. [2, Exercise 8.2] Let Γ be a finite and acyclic quiver and let $\Lambda = k\Gamma$. The aim of this exercise is to show that $\operatorname{rad} \Lambda = R_\Gamma$ where R_Γ is the ideal generated by the set of arrows of Γ.

   (a) Let $\alpha \in \Gamma_1$, i.e. α is an arrow in Γ. Show that $\alpha \in \operatorname{rad} \Lambda$.

   (b) Show that Λ/R_Γ is a semisimple Λ-module.

   (c) Use the equality $\operatorname{rad} \Lambda = \bigcap_{S \text{ simple}} \operatorname{Ann}_\Lambda S$ and part (b) to deduce that $\operatorname{rad} k\Gamma \subseteq R_\Gamma$.

   (d) Combining (a) and (c) we get that $\operatorname{rad} k\Gamma = R_\Gamma$.

3. Is it true that $\operatorname{rad} k\Gamma = R_\Gamma$ for any finite quiver Γ?

4. [1, Lemma 1.3](Challenge) Let Λ be a ring. In this exercise we give some more equivalent characterizations of $\operatorname{rad} \Lambda$.

   (a) Show that $\lambda \in \operatorname{rad} \Lambda$ if and only if for any $x \in \Lambda$ the element $1 - x\lambda$ has a *two-sided inverse*, i.e. an element $y \in \Lambda$ such that $y(1 - x\lambda) = 1 = (1 - x\lambda)y$.

   (b) Show that $\lambda \in \operatorname{rad} \Lambda$ if and only if $1 - \lambda x$ has a two-sided inverse for any $x \in \Lambda$.

5. [1, Lemma 1.3] Let Λ be a ring. Recall that the *opposite ring* of Λ, denoted Λ^{op}, is defined to be the ring with the same underlying set and group structure as Λ, but where the multiplication in Λ^{op} is defined by the formula $a * b = ba$.

(a) Show that a subset of Λ is a right ideal of Λ if and only if it is a left ideal in Λ^{op}. Deduce that $\operatorname{rad}\Lambda^{\mathrm{op}}$ is equal to the intersection of all maximal right ideals in Λ.

(b) Show that the following are equivalent for $\lambda \in \Lambda$:

i. $\lambda \in \operatorname{rad}\Lambda^{\mathrm{op}}$

ii. For any $x \in \Lambda$ the element $1 - \lambda x$ has a right inverse

iii. For any $x \in \Lambda$ the element $1 - \lambda x$ has a two-sided inverse

(c) Using part (b) of the Exercise 3, show that $\operatorname{rad}\Lambda^{\mathrm{op}} = \operatorname{rad}\Lambda$.

6. Let $f : \Lambda \to \Lambda'$ be a morphism between k-algebras, and assume all simple left Λ'-modules have dimension 1 over k. Show that $f(\operatorname{rad}\Lambda) \subseteq \operatorname{rad}\Lambda'$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, `https://wiki.math.ntnu.no/ma3203/2021v/course_schedule`.