MA3203 - Exercise sheet 8

Throughout Λ denotes a ring.

1. (a) Show that the collection of Noetherian Λ-modules is closed under extensions.
   (b) Deduce that a Λ-module of finite length must be Noetherian.

2. Let M be a semisimple Λ-module. Show that the following are equivalent:
   (a) M is Artinian.
   (b) M is Noetherian.
   (c) M has finite length.

3. Let M be a Λ-module
   (a) Show that if M is Noetherian, then M is finitely generated.
   (b) Let Λ be a left Artinian ring, and assume M is finitely generated. Show that M has finite length.
   (c) Deduce that if Λ is left Artinian, then the following are equivalent:
      (i) M has finite length.
      (ii) M is Noetherian.
      (iii) M is finitely generated.

      (It is also true that this is equivalent to M being Artinian, but this is a bit more difficult to show).

4. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of Λ-modules.
   (a) Show that M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.
   (b) Show that M_2 is Artinian if and only if M_1 and M_3 are Artinian.