MA3203 - Exercise sheet 17

Throughout k denote a field

1. [2, Problem 4.1 and 4.2] Let Σ be a k-algebra.

   (a) Let Λ be a k-algebra and let $\mathcal{B}$ be a basis for Λ. Supposed $\varphi\colon \Lambda \to \Sigma$ is a morphism of k-vector spaces satisfying the following

   - $\varphi(1_\Lambda) = 1_\Sigma$.
   - $\varphi(b_1 b_2) = \varphi(b_1)\varphi(b_2)$ for all $b_1, b_2 \in \mathcal{B}$.

   Show that φ is a morphism of k-algebras

   (b) Let Γ be a quiver and let $f\colon \Gamma_0 \cup \Gamma_1 \to \Sigma$ be a function satisfying the following:

   - $\sum_{v \in \Gamma_0} f(v) = 1_\Sigma$.
   - $qr = 0$ (concatenation of paths) implies that $f(q)f(r) = 0$ for all $q, r \in \Gamma_0 \cup \Gamma_1$.
   - $f(v) = f(v)^2$ for all $v \in \Gamma_0$.
   - $f(e(\alpha))f(\alpha) = f(\alpha) = f(\alpha)f(s(\alpha))$ for all $\alpha \in \Gamma_1$, where $e(\alpha)$ and $s(\alpha)$ denotes the end and start of α, respectively

   Then f extends uniquely to a morphism of k-algebras $\tilde{f}\colon k\Gamma \to \Sigma$.

2. (Used in the end of the proof of Theorem 50 in the lecture)

   Let Λ be a finite-dimensional algebra, and let r be the Jacobson radical of Λ. Let $x_1, \cdots x_n$ be elements of r, and assume their image $\overline{x_1}, \cdots \overline{x_n}$ in r/r^2 generate r/r^2 as a Λ/r-module. Show that $x_1, \cdots x_n$ generate r as a Λ-module

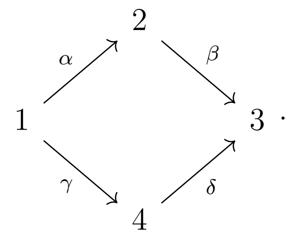
3. [1, Exercise II.16] This exercise shows that the assumption that k is algebraically closed is necessary in order to claim that every basic k-algebra is isomorphic to a quiver algebra.

(a) Show that $\mathbb{C}$ is a two-dimensional basic and connected $\mathbb{R}$-algebra.

(b) Show that there is no quiver Q and admissible ideal I so that $\mathbb{C} \cong \mathbb{R}Q/I$.

4. [1, Exercise II.17] Here we show that the generators of an admissible ideal are not uniquely determined in general:

   (a) Let Γ be the quiver



   and let $\mathcal{I}_1 = \langle \beta\alpha + \delta\gamma \rangle$ and $\mathcal{I}_2 = \langle \beta\alpha - \delta\gamma \rangle$ be two ideals of $k\Gamma$. If $\mathsf{char} k \neq 2$, show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are admissible and distinct, and that there is an k-algebra isomorphism $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$.

   (b) Let Γ be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \xrightarrow{\gamma} 3$ and let $\mathcal{I}_1 = \langle \gamma\alpha - \gamma\beta \rangle$ and $\mathcal{I}_2 = \langle \gamma\alpha \rangle$ be two ideals of $k\Gamma$. Show that $\mathcal{I}_1$ and $\mathcal{I}_2$ are admissible and distinct, and that there is an k-algebra isomorphism $k\Gamma/\mathcal{I}_1 \cong k\Gamma/\mathcal{I}_2$ (Here the characteristic of k can be arbitrary).

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).

[2] Ø. Solberg, 2017 MA3203 Problem Sheets, NTNU, `http://wiki.math.ntnu.no/ma3203/2017v/ovinger`.