MA3203 - Exercise sheet 15

Most of Problem 1-6 and 9 is taken from Problem set 14 in [1]. Throughout k denotes a field.

1. *The categories of Rings and Sets.*

   (a) Show that there is a well-defined category Set whose objects are sets, morphisms are the usual maps between sets, and composition is the usual composition.

   (b) Show that there is a well-defined category Ring whose objects are rings (with $1 \neq 0$), morphisms are ring homomorphisms (sending 1 to 1), and composition is the usual composition.

   (c) Show that there is a forgetful functor $F\colon \text{Ring} \to \text{Set}$ sending a ring to its underlying set, and a morphism of rings to a map of the underlying sets.

2. *Monomorphisms and epimorphisms.*

   Let $\mathcal{C}$ be a category and let A, B be objects of $\mathcal{C}$. A morphism $f \in \text{Hom}_{\mathcal{C}}(A, B)$ is called a *monomorphism* if for any object $C \in \mathcal{C}$ and any pair of morphisms $g_1, g_2 \in \text{Hom}_{\mathcal{C}}(C, A)$ we have $f \circ g_1 = f \circ g_2$ if and only if $g_1 = g_2$. Likewise, f is called an *epimorphism* if for any object $C \in \mathcal{C}$ and any pair of morphisms $g_1, g_2 \in \text{Hom}_{\mathcal{C}}(B, C)$ we have $g_1 \circ f = g_2 \circ f$ if and only if $g_1 = g_2$.

   (a) Let $\mathcal{C}$ be arbitrary. Show that any isomorphism in $\mathcal{C}$ is both a monomorphism and an epimorphism.

   (b) Let f be a morphism in Set or in $\text{Mod}\,\Lambda$ for a ring Λ. Show that

      (i) f is a monomorphism if and only if it is injective.

      (ii) f is an epimorphism if and only if it is surjective.

(iii) f is an isomorphism if and only if it is both a monomorphism and an epimorphism.

(c) Show that the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring, but not an isomorphism.

(d) (Challenge) Show that every monomorphism in Ring is injective (but note that we have already shown that not every epimorphism in Ring is surjective!).

(e) Show that $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$ in the category Set.

3. *Groups as categories*

Let G be a group and let $\{*\}$ be a set with one element. Define a category $\mathcal{G}$ with a single object $*$, morphisms $\mathrm{Hom}_{\mathcal{G}}(*, *) = G$, and composition given by $g \circ h := gh$ for all $g, h \in \mathrm{Hom}_{\mathcal{G}}(*, *)$. Show that $\mathcal{G}$ is a well-defined category and that every $g \in \mathrm{Hom}_{\mathcal{G}}(*, *)$ is an isomorphism.

4. *Initial and terminal objects.*

Let $\mathcal{C}$ be a category. An object I of $\mathcal{C}$ is called *initial* if for any object $A \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(I, A)$ contains a single element. Likewise, an object T of $\mathcal{C}$ is called *terminal* if for any object $A \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(A, T)$ contains a single element. If the same object is both initial and terminal, it is called a *zero object*.

(i) Show that the trivial group is a zero object in the category Ab (and more generally in $\mathrm{Mod}\,\Lambda$ for any ring Λ).

(ii) Show that $\mathbb{Z}$ is initial in Ring.

(iii) (Challenge) Show that if A and B are both initial (resp. both terminal) in some category $\mathcal{C}$, then A and B are isomorphic.

5. *Posets*

(a) Recall that a poset P is a set S together with a reflexive, antisymmetric, transitive relation $\leq$ (meaning that (1) for all $x, y, z \in P$, we have $x \leq x$ (2) if $x \leq y$ and $y \leq x$, then $x = y$, and (3) if $x \leq y$ and $y \leq z$ then $x \leq z$). An order-preserving map of posets $P \to Q$ is a map $f : P \to Q$ of sets so that if $x \leq y$ in P then $f(x) \leq f(y)$ in Q. Show that there is a well-defined category Pos whose objects are posets, morphisms are order-preserving maps, and composition is the usual composition.

(b) Let P be a poset and let $\{*\}$ be a set with one element. Show that there is a well-defined category $\mathcal{P}$ whose objects are the elements of P and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{P}}(x,y) = \begin{cases} \{*\} & x \leq y \\ \emptyset & x \not\leq y. \end{cases}$$

What is the composition law in this category? What are the identity morphisms? Which morphisms are isomorphisms?

6. *The Hom-functor.*

Let $\mathcal{C}$ be any category and let A be an object in $\mathcal{C}$.

(a) Show that there is a covariant functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Set}$ which sends an object B of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and which sends a morphism $f : B \to C$ in $\mathcal{C}$ to the map $f_* : \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$ given by $f_*(g) := f \circ g$.

(b) Show that there is a contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,A) : \mathcal{C} \to \mathrm{Set}$ which sends an object B of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(B,A)$ and which sends a morphism $f : B \to C$ in $\mathcal{C}$ to the map $f^* : \mathrm{Hom}_{\mathcal{C}}(C,A) \to \mathrm{Hom}_{\mathcal{C}}(B,A)$ given by $f^*(g) := g \circ f$.

(c) Assume $\mathcal{C}$ is a R-category, where R is a commutative ring. Show that $\mathrm{Hom}_{\mathcal{C}}(A,-)$ as in (a) defines a covariant R-functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Mod}\,R$. Dually, show that $\mathrm{Hom}_{\mathcal{C}}(-,A)$ as in (b) defines a contravariant R-functor $\mathrm{Hom}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Mod}\,R$

7. *The functor category*

(a) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Show that there exists a category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are morphisms of functors.

(b) Let Γ be a quiver, considered as a category. Show that $\mathrm{Fun}(\Gamma, \mathrm{Vec}(k))$ is equivalent to the category of representations of Γ in k.

8. *The opposite category*

Let $\mathcal{C}$ be a category.

(a) Show that there exists a category, denoted $\mathcal{C}^{\text{op}}$, and given as follows

- The objects of $\mathcal{C}^{\text{op}}$ are the same as the objects of $\mathcal{C}$.
- A morphism $f\colon C \to D$ in $\mathcal{C}^{\text{op}}$ is the same as a morphism $f\colon D \to C$ in $\mathcal{C}$
- A composite of morphism $g \circ f$ in $\mathcal{C}^{\text{op}}$ is defined to be the composite $f \circ g$ in $\mathcal{C}$

(b) Show that $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$.

(c) Let $\mathcal{D}$ be a category. Show that a contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F\colon \mathcal{C}^{\text{op}} \to \mathcal{D}$.

(d) Let $\text{vec}(k)$ denote the category of finite-dimensional vector spaces. Show that the contravariant functor $\text{Hom}_k(-,k)\colon \text{vec}(k) \to \text{vec}(k)$ gives an equivalence $\text{vec}(k)^{\text{op}} \xrightarrow{\cong} \text{vec}(k)$.

9. *Left exact functors*

Let Λ be a ring and let $\text{Mod}\,\Lambda$ be the category of left Λ-modules. A covariant functor $F : \text{Mod}\,\Lambda \to \text{Ab}$ is called *left exact* if for any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of Λ-modules the sequence

$$0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is exact (this means that $F(f)$ is injective and $\ker F(g) = \text{Im}F(f)$). If in addition $F(g)$ is surjective, then F is called *exact*.

Let M be a left Λ-module. Show that $\text{Hom}_\Lambda(M,-) : \text{Mod}\,\Lambda \to \text{Ab}$ is always left exact and is exact if and only if M is projective.

10. *Morita equivalence*

Let Λ_1 and Λ_2 be rings. We say that Λ_1 and Λ_2 are *Morita equivalent* if the module categories $\text{Mod}\,\Lambda_1$ and $\text{Mod}\,\Lambda_2$ are equivalent as $\mathbb{Z}$-categories.

(a) Let $F\colon \text{Mod}\,\Lambda_1 \to \text{Mod}\,\Lambda_2$ be an equivalence of $\mathbb{Z}$-categories, and let M be a Λ_1-module. Show the following:

(i) F preserves monomorphisms and epimorphisms.

(ii) $M \neq 0$ if and only if $F(M) \neq 0$.

(iii) M is a simple Λ_1-module if and only $F(M)$ is a simple Λ_2-module

(iv) M is a projective Λ_1-module if and only $F(M)$ is an projective Λ_2-module

(v) M is an indecomposable Λ_1-module if and only $F(M)$ is an indecomposable Λ_2-module

(b) Let $M_n(k)$ be the ring of $n \times n$ matrices over the field k, and let

$$F\colon \operatorname{Mod} M_n(k) \to \operatorname{Mod}(k)$$

be the forgetful functor which sends a $M_n(k)$-module to its underlying vector space and a morphism of $M_n(k)$-module to the underlying morphism of vector spaces. Show that F is *not* an equivalence.

(c) (Challenge) Show that $M_n(k)$ and k are Morita equivalent.

References

[1] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.