MA3203 - Exercise sheet 1

Throughout k denotes a field and $\Gamma = (\Gamma_0, \Gamma_1)$ denotes a finite quiver.

1. (a) Let Γ be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, and let $R = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$ be a lower triangular matrix ring. Show that the map

$$\phi\colon k\Gamma \to R$$

given by

$$\phi(e_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \phi(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \phi(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi(\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \phi(\beta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \phi(\beta\alpha) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is an isomorphism of algebras.

   (b) Let R denote the lower triangular matrix ring $\begin{bmatrix} k & 0 & \cdots & 0 \\ k & k & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ k & k & \cdots & k \end{bmatrix}$ with n-rows and n-columns. Show that R is isomorphic to the path algebra of the quiver $1 \to 2 \to \cdots \to n-1 \to n$

2. Let V be a finite-dimensional vector space over k.

   (a) Show that the set

$$\begin{bmatrix} k & 0 \\ V & k \end{bmatrix} := \{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, c \in k \text{ and } b \in V \}$$

has a natural structure of a k-algebra.

(b) For $V = k^2$, show that $\begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix}$ is isomorphic to the path algebra of the quiver $1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$.

(c) For V arbitrary, can you find a quiver Γ such that $\begin{bmatrix} k & 0 \\ V & k \end{bmatrix}$ is isomorphic to $k\Gamma$?

We say that a quiver Γ is connected if for any two vertices there exists an *undirected* path between them. We say that an algebra A is *connected* if it cannot be written as product $A_1 \times A_2$ of two algebras A_1 and A_2.

3 In this exercise we prove Lemma 1.7 in [1], i.e. that Γ is connected if and only if $k\Gamma$ is connected.

(a) Let Γ and Γ' be two quivers, and let $\Gamma \cup \Gamma'$ denote their disjoint union. Show that $k\Gamma \times k\Gamma'$ and $k(\Gamma \cup \Gamma')$ are isomorphic as k-algebras.

(b) Deduce that if $k\Gamma$ is connected as a k-algebra, then Γ is connected as a quiver

(c) (Challenge) Assume $k\Gamma = R_1 \times R_2$ for two k-algebras R_1 and R_2. Let e_i be the trivial path at a vertex $i \in \Gamma_0$. Show that $e_i \in R_1$ or $e_i \in R_2$.

(d) Show that if Γ is connected, then $k\Gamma$ is connected. (Hint: use (c)).

Recall that a ring R is *local* if it has a unique maximal left ideal.

4 [1, Exercise II.2cd] Let Γ be a connected quiver.

(a) Show that $k\Gamma$ is a local ring if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| = 0$.

(b) Show that $k\Gamma$ is commutative if and only if $|\Gamma_0| = 1$ and $|\Gamma_1| \leq 1$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).