List of some of the main definitions and results from MA3203

1. *Part 1: Quivers and path algebras*

   - Definition of a quiver (what is a path?)
   - Path algebra of a quiver (how do you multiply two elements in $k\Gamma$, and what is the identity element in $k\Gamma$).
   - Definition of k-algebras. Why is $k\Gamma$ a k-algebra?
   - Proposition 1: $k\Gamma$ is finite dimensional if and only if Γ has no oriented cycles
   - Proposition 2: If $\Gamma = (\Gamma_0, \Gamma_1)$ has no oriented cycles, then $k\Gamma$ is semisimple if and only if $\Gamma_1 = \emptyset$.
   - Proposition 3: If J consists of all linear combinations of all non-trivial paths of $k\Gamma$, then J is an ideal of $k\Gamma$ and $k\Gamma/J \cong k \times \cdots \times k$ where the product is taken $|\Gamma_0|$ times.
   - Γ is a connected quiver if and only if $k\Gamma$ is a connected k-algebra (Given as a exercise, not necessary to know the proof).

2. *Part 2: Representations of quivers and homomorphisms of such*

   - Quiver representations + examples (what is the zero representation, what are the simple representations, what is an indecomposable representation, what is a subrepresentation, what is a morphism between representations, what is a kernel of a morphism between representations, what is a image of a morphism between representations)
   - The correspondence between modules over the path algebra $k\Gamma$ and representations of the quiver Γ.

- Definition of indecomposable modules. Definition of finite representation type of an algebra
- Theorem 4: k field of characteristic p and G a finite group such that p divides $|G|$. Then the group algebra kG is of finite representation type if and only if all Sylow subgroups of G are cyclic.
- Theorem 5: Γ connected quiver without oriented cycles. Then $k\Gamma$ is of finite representation type if and only if the underlying graph of Γ is a Dynkin diagram.

3. *Part 3: Quivers with relations, quotients of path algebras and modules over such*

   - Definition of a relation in a quiver, definition of quivers with relations. Definition of an admissible relation.
   - Correspondence between representations of a quiver Γ with relations $\rho = \{\sigma_l\}_{l \in I}$ and modules over the algebra $k\Gamma/(\rho)$.

4. *Part 4: Modules of finite length and Jordan-Hölder theorem*

   - Generalized composition series, composition series, composition factors, finite length of a module, + examples. $m_S^{\mathcal{F}}(A)$ and $l_{\mathcal{F}}(A)$ for a module A, a simple module S, and a composition series $\mathcal{F}$.
   - Short exact sequences of modules. Short exact sequences of representations.
   - Given a short exact sequence $0 \to A \to B \to C \to 0$, and a generalized composition series $\mathcal{F}$ for B, can get induced generalized composition series $\mathcal{F}'$ and $\mathcal{F}''$ of A and C (Proposition 7).
   - Theorem 9 (Jordan–Hölder): If B is a module of finite length and $\mathcal{F}$ and $\mathcal{G}$ are generalized composition series of B, then $m_S^{\mathcal{F}}(B) = m_S^{\mathcal{G}}(B)$ and $l_{\mathcal{F}}(B) = l_{\mathcal{G}}(B)$
   - Proposition 11: If $f\colon A \to A$ is an endomorphism of a module A of finite length, then f is an isomorphism if and only if it is a monomorphism if and only if it is an epimorphism.
   - Proposition 12: If $0 \to A \to B \to C \to 0$ is exact and A and C have finite length, then B has finite length and $l(A)+l(C) = l(B)$.

- Proposition 13: The subcategory of finite length modules is closed under extensions, and is the smallest subcategory closed under extensions and containing the simple modules.
- Proposition 14: A module is of finite length if and only if it is Noetherian and Artinian
- A ring Λ is left artinian if and only if Λ has finite length as a left Λ-module
- For a left artinian ring Λ, the modules of finite length coincide with the finitely generated modules.

5. *Part 5: The radical of rings and modules*

- Jacobson radical of a ring + examples
- Proposition 16 + Exercise: Λ ring, $\lambda \in \Lambda$. The following are equivalent:
  - $\lambda \in \operatorname{rad} \Lambda$
  - $1 - x\lambda$ is left invertible for all $x \in \Lambda$
  - $1 - x\lambda$ has a two-sided inverse for all $x \in \Lambda$
  - $1 - \lambda x$ has a two-sided inverse for all $x \in \Lambda$
  - $\lambda \cdot S = (0)$ for all simple left Λ-modules S

  (Not necessary to remember the proof)
- $\operatorname{rad} \Lambda = \operatorname{rad} \Lambda^{\mathrm{op}}$.
- Corollary 17: $\operatorname{rad} \Lambda = \bigcap_{S \text{ simple left } \Lambda \text{ module}} \operatorname{Ann}_\Lambda S$. In particular, $\operatorname{rad} \Lambda$ is a two-sided ideal.
- Theorem 18, Nakayama's lemma: M finitely generated Λ-module, I ideal of Λ contained in $\operatorname{rad} \Lambda$. If $I \cdot M = M$, then $M = 0$.
- Lemma 19: If Λ is left artinian, then $\operatorname{rad} \Lambda$ is a nilpotent ideal. Furthermore, $\operatorname{rad} \Lambda$ is the largest nilpotent left ideal of Λ.
- Theorem 20: A ring Λ is semisimple if and only if it is left artinian and $\operatorname{rad} \Lambda = 0$.
- Theorem 21: Λ left artinian ring. Then
  - $\Lambda / \operatorname{rad} \Lambda$ is a semisimple ring
  - A left module M is semisimple if and only if $\operatorname{rad} \Lambda \cdot M = 0$

  - There are only finitely many simple left Λ-modules, and they occur as direct summands of $\Lambda/\operatorname{rad}\Lambda$.
- Corollary 22: The following are equivalent:
  - Λ is left artinian
  - Every finitely generated Λ-module has finite length
  - $\operatorname{rad}\Lambda$ is nilpotent and $\operatorname{rad}(\Lambda)^i/\operatorname{rad}(\Lambda)^{i+1}$ is a finitely generated semimsimple Λ-module for all $i \geq 0$.
- Theorem 23: Λ left artinian and I a nilpotent left ideal of Λ. Then $I = \operatorname{rad}\Lambda$ if and only if Λ/I is semisimple.
- Proposition 24: Let (Γ, ρ) be a quiver with admissible relations. Then $\operatorname{rad}(k\Gamma/(\rho)) = J/\rho = \overline{J}$ where J is the ideal in $k\Gamma$ generated by the arrows.
- Small submodule, radical of a module.
- Proposition 25: B finitely generated Λ-module, $A \subseteq B$ is small if and only if $A \subseteq \operatorname{rad} B$.
- Theorem 26: Λ left artinian ring, A finitely generated Λ-module. Then $\operatorname{rad} A = \operatorname{rad}\Lambda \cdot A$.
- The radical of a representation + examples
- The top of A, defined as $A/\operatorname{rad} A$ for a module A over a left artinian ring Λ
- Λ left artinian, $f\colon A \to B$ morphism of finitely generated Λ-modules. Then f is surjective if and only if the induced map $\overline{f}\colon A/\operatorname{rad} A \to B/\operatorname{rad} B$ is surjective.
- Essential epimorphism, examples, composition of essential epimorphisms is an essential epimorphism.
- Proposition 28: Λ left artinan, $f\colon A \to B$ surjective map of finitely generated Λ-modules. The following are equivalent:
  - f is an essential epimorphism
  - $\operatorname{Ker} f \subseteq \operatorname{rad} A$
  - $\overline{f}\colon A/\operatorname{rad} A \to B/\operatorname{rad} B$ is an isomorphism

6. *Part 6: Projective modules*

- Projective modules, free modules are projective,

- Proposition 29: A module is projective if and only if it is a direct summand of a free module.
- e idempotent of Λ, then Λe is a projective left Λ-module
- Projective cover of a module, examples.
- Theorem 30: Λ left artinian ring. Then any finitely generated Λ-module has a projective cover, which is unique up to isomorphism
- Proposition 31: Λ left artinian, $f\colon P \to A$ surjective morphism of Λ-modules with P projective. Then f is a projective cover if and only if $\overline{f}\colon P/\operatorname{rad} P \to A/\operatorname{rad} A$ is an isomorphism.
- Proposition 31: Λ left artinian, then the map $f\colon P_1 \oplus P_2 \oplus \cdots \oplus P_n \to A_1 \oplus A_2 \oplus \cdots \oplus A_n$ defined by

  $$(p_1, p_2, \cdots, p_n) \mapsto (f_1(p_1), f_2(p_2), \cdots, f_n(p_n))$$

  is a projective cover if and only if each $f_i\colon P_i \to A_i$ is a projective cover
- Proposition 32: The following hold for a left artinian ring Λ and a finitely generated projective module P :
  - $P \to P/\operatorname{rad} P$ is a projective cover
  - Q finitely generated projective module, then $P \cong Q$ if and only if $P/\operatorname{rad} P \cong Q/\operatorname{rad} Q$.
  - P is indecomposable if and only if $P/\operatorname{rad} P$ is simple
  - $P = \bigoplus_{i=1}^n P_i \cong \bigoplus_{j=1}^m Q_j$ with P_i, Q_j indecomposable, then $m = n$ and exists permutation π of $\{1, 2, \cdots, m\}$ such that $P_i \cong Q_{\pi(i)}$ for all $1 \leq i \leq n$.
- Corollary 33:Λ left artinian, and let $S_1, S_2, \cdots, S_n$ be the simple Λ-modules. Let P_i be the projective cover of S_i. Then the indecomposable Λ-modules are up to isomorphism just the modules $P_1, \cdots, P_n$
- Lemma 34: Λ left artinian, $f\colon P \to M$ morphism of finitely generated Λ-modules. Assume the composite $P \xrightarrow{f} M \xrightarrow{\pi_m} M/\operatorname{rad} M$ is a projective cover. Then f is a projective cover.
- Local rings, examples for finite-dimensional algebras.
- Proposition 35: Λ local, then 0 and 1 are the only idempotents

- Proposition 37: Λ left artinian, P finitely generated Λ-module. The following are equivalent:
  - P is indecomposable
  - $\mathrm{End}_\Lambda(P)$ is local
  - $\mathrm{rad}\, P$ is the unique maximal submodule of P
  - $P/\mathrm{rad}\, P$ is simple
- Corollary 38: Λ left artinian. The following are equivalent:
  - Λ local
  - $\mathrm{rad}\,\Lambda$ unique maximal left ideal
  - $\Lambda/\mathrm{rad}\,\Lambda$ is simple
- Proposition 39: Λ left artinian. The following hold:
  - the identity $1 \in \Lambda$ can be written as a sum $1 = e_1 + e_2 + \cdots + e_n$ of primitive orthogonal idempotents
  - $e_1, e_2, \cdots e_n$ orthogonal idempotents, and $e = e_1 + e_2 + \cdots + e_n$, then $\Lambda e = \Lambda e_1 \oplus \Lambda e_2 \oplus \cdots \oplus \Lambda e_n$
  - $e \neq 0$ idempotent. Then e is primitive if and only if Λe is indecomposable
- Proposition 40: Λ left artinian. Any finitely generated projective module can be written as a sum of indecomposable projective modules in a unique way up to isomorphism and permutation of the summands.

7. *Part 7: Krull-Remak-Schmidt theorem*

- Lemma 41 (Fitting lemma): Λ ring, M a Λ-module of finite length, $\phi\colon M \to M$ morphism of Λ-modules. Then there exists an $n \geq 1$ such that $M = \mathrm{Im}\,\phi^n \oplus \mathrm{Ker}\,\phi^n$.
- Theorem 42: M finitely generated Λ-module where Λ is a left artinian ring. Then M is indecomposable if and only if $\mathrm{End}_\Lambda(M)$ is a local ring.
- Theorem 43 (Krull–Remak–Schmidt theorem): Λ left artinian ring, M finitely generated Λ-module. Then M can be written as a sum of indecomposable Λ-modules in a unique way up to isomorphism and permuation of the summands.

8. *Part 8: Artin algebras*

   - R-algebras for R a commutative ring
   - Definition of an artin R-algebra. Any artin R-algebra is a left artinian ring (Proposition 44 c))
   - Proposition 44 a) and b): Λ artin R-algebra. Then $\mathrm{Hom}_\Lambda(A, B)$ is a finitely generated R-module for any finitely generated left Λ-modules A and B. Also $\mathrm{End}_\Lambda(A)$ is an artin R-algebra for any finitely generated left Λ-module A.

9. *Part 9: Categories and functors*

   - What is a category, examples of categories, isomorphisms in categories, subcategories, full subcategories + examples
   - Covariant and contravariant functors + examples (Hom-functor)
   - preadditive R-category for a commutative ring R. Additive R-functors + examples
   - Morphism of functors (also called natural transformations). Examples
   - Equivalences of (R) categories. A functor is an equivalence if and only if it is full, faithful and dense (Proposition 45).
   - Theorem 46: (Γ, ρ) quiver with admissible relations. Then the category $\mathrm{Rep}(\Gamma, \rho)$ is equivalent to $\mathrm{mod}\,\Lambda$, where $\Lambda = k\Gamma/(\rho)$. Description of this equivalence.

10. *Part 10: Projectivization*

   - Lemma 47: Have isomorphism

     $$\mathrm{Hom}_\Lambda(A, B_1 \oplus B_2) \cong \mathrm{Hom}_\Lambda(A, B_1) \oplus \mathrm{Hom}_\Lambda(A, B_2)$$

     of left $\mathrm{End}_\Lambda(A)^{\mathrm{op}}$-modules. Dually, have isomorphism

     $$\mathrm{Hom}_\Lambda(A_1 \oplus A_2, B) \cong \mathrm{Hom}_\Lambda(A_1, B) \oplus \mathrm{Hom}_\Lambda(A_2, B)$$

     of left $\mathrm{End}_\Lambda(B)$-modules

- Proposition 48: Λ artin R-algebra, A finitely generated Λ-module, $\Gamma = \mathrm{End}_\Lambda(A)^{\mathrm{op}}$. Then $e_A(X)$ is a projective Γ-module, and

$$e_A\colon \operatorname{add} A \to \mathcal{P}(\Gamma)$$

is an equivalence of R-categories, where $\mathcal{P}(\Gamma)$ is the subcategory of $\operatorname{mod}\Gamma$ of finitely generated projective Γ-modules.

- Lemma 49: Same assumptions as in Proposition 48. The following hold:
  (a) $X \neq 0$ in $\operatorname{add} A$ if and only if $e_A(X) \neq 0$ in $\mathcal{P}(\Gamma)$.
  (b) $X \in \operatorname{add} A$, then X is indecomposable if and only if $e_A(X)$ is indecomposable.
  (c) $X, Y \in \operatorname{add} A$, then $e_A(X) \cong e_A(Y)$ if and only if $X \cong Y$.

11. *Part 11: Basic artin algebras*

- Basic artin algebras, examples and non-examples
- Proposition 49: Any artin algebra is Morita equivalent to a basic artin algebra (for more details see the lectures).
- Theorem 50: Any basic finite-dimensional algebra over an algebraically closed field k is isomorphic to an algebra of the form $k\Gamma/(\rho)$ where Γ is a quiver with admissible relations ρ.

12. *Part 12: Duality*

- Duality of categories.
- $D = \mathrm{Hom}_k(-, k)$ form a duality on the category of finite-dimensional k-vector spaces. Description of the isomorphism of functors $\mathrm{Id}_{\mathrm{vec}(k)} \to DD$.
- Proposition 51: For a finite-dimensional algebra Λ the functor $D = \mathrm{Hom}_k(-, k)$ extends to a well-defined contravariant functor

$$D\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$$

which is a duality

- Description of D as a functor on the category of representations of a quiver with admissible relations.

- Lemma 52: The functor D preserves short exact sequences, simple modules, and the length of any finitely generated module

13. *Part 13: Injective modules*

    - Injective module, essential submodule, injective envelope, socle of a module.
    - Proposition 53: Λ finite-dimensional algebra, $P \in \operatorname{mod}\Lambda$. Then P is projective if and only if $D(P)$ is injective
    - Proposition 53: Any $M \in \operatorname{mod}\Lambda$ is a submodule of a projective module
    - Lemma 54: Λ artin R-algebra, $X \in \operatorname{mod}\Lambda$, and A submodule of X. Then A is an essential submodule of X if and only if $\operatorname{soc} X \subseteq A$ if and only if $\operatorname{soc} X = \operatorname{soc} A$
    - Proposition 55: Λ artin R-algebra, $(0) \neq A \in \operatorname{mod}\Lambda$. The following hold
        - A map $A \to I$ is an injective envelope if it is a monomorphism, I is injective, and $\operatorname{soc} I = \operatorname{soc} A$.
        - Injective envelopes are unique up to isomorphism
        - If I is injective and $A \to I$ is a morphism such that the restriction $\operatorname{soc} A \to I$ is an injective envelope, then $A \to I$ is an injective envelope.
    - Lemma 56: $A \in \operatorname{mod}\Lambda$ where Λ is an artin R-algebra. Then $\operatorname{soc} A = \{a \in A \mid \operatorname{rad}\Lambda \cdot a = (0)\}$.
    - $\operatorname{soc} A \cong \operatorname{Hom}_\Lambda(\Lambda/\operatorname{rad}\Lambda, A)$
    - The duality $D(-)$ gives a bijection between projective covers in $\operatorname{mod}\Lambda$ and injective envelopes in $\operatorname{mod}\Lambda^{\mathrm{op}}$.
    - Λ finite-dimensional algebra. There is a bijection between isomorphism classes of simple Λ-modules and isomorphism classes of indecomposable injective Λ-modules.
    - Socle of a representation.