

MA3203 - Exercise sheet 7

Throughout Λ denotes a ring.

1. Let S and S' be simple Λ -modules, and let $f: S \rightarrow S'$ be a non-zero morphism of Λ -modules. Show that f is an isomorphism.
2. A sequence of morphisms of Λ -modules of the following form

$$0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \rightarrow 0$$

is called *exact* if f_1 is a monomorphism, f_n is an epimorphism, and $\text{Im } f_i = \ker f_{i+1}$ for all $1 \leq i \leq n-1$.

- (a) Show that $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \rightarrow 0$ is exact if and only if there exist short exact sequences

$$\begin{aligned} 0 \rightarrow M_1 &\xrightarrow{f_1} M_2 \xrightarrow{g_2} K_2 \rightarrow 0 \\ 0 \rightarrow K_2 &\xrightarrow{h_2} M_3 \xrightarrow{g_3} K_3 \rightarrow 0 \\ &\vdots \\ 0 \rightarrow K_{n-2} &\xrightarrow{h_{n-2}} M_{n-1} \xrightarrow{g_{n-1}} K_{n-1} \rightarrow 0 \\ 0 \rightarrow K_{n-1} &\xrightarrow{h_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \rightarrow 0 \end{aligned}$$

such that $f_i = h_i \circ g_i$ for $2 \leq i \leq n-1$

- (b) Deduce that if $0 \rightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \rightarrow 0$ is exact and M_i has finite length for all i , then

$$\sum_{i=1}^{n+1} (-1)^{i-1} l(M_i) = 0$$

3. [1, Exercise 4.2] Let M be a Λ -module of finite length, and let K, L be submodules of M . Show that $\ell(K + L) = \ell(K) + \ell(L) - \ell(K \cap L)$.
(This generalizes the formula $\dim(V+W) = \dim V + \dim W - \dim V \cap W$ for vector spaces.)
4. (a) Show that the collection of finitely generated Λ -modules is closed under extensions.
(b) Deduce that a Λ -module of finite length must be a finitely generated Λ -module.

References

- [1] L.-P. Thibault, 2019 MA3203 Problem Sheets, NTNU, http://wiki.math.ntnu.no/ma3203/2019v/lecture_plan.