

MA3203 - Exercise sheet 13

Throughout k denotes a field

1. [2, Exercise 12.1] Let $\Gamma = 1 \rightrightarrows 2$. For $\lambda \in k$ define a representation $M(\lambda) = k \begin{smallmatrix} \xrightarrow{1} \\ \xrightarrow{\lambda} \end{smallmatrix} k$ and define $M(\infty) = k \begin{smallmatrix} \xrightarrow{0} \\ \xrightarrow{1} \end{smallmatrix} k$. For all $\lambda \in k \cup \{\infty\}$, show that there is a projective cover $\Lambda e_1 \xrightarrow{f_\lambda} M(\lambda)$ and find $\ker(f_\lambda)$.

2. Let Λ be a ring, and let e be an idempotent of Λ .

- (a) Let M be a Λ -module. Show that we have an isomorphism

$$\mathrm{Hom}_\Lambda(\Lambda e, M) \cong eM$$

of abelian groups.

- (b) Show that $e\Lambda e$ inherits a ring structure from Λ with identity element e . Furthermore, show that with this structure

$$\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)^{\mathrm{op}} \cong e\Lambda e$$

becomes an isomorphism of rings, where $\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)^{\mathrm{op}}$ denotes the opposite ring of the endomorphism ring $\mathrm{Hom}_\Lambda(\Lambda e, \Lambda e)$. Also, show that

$$\mathrm{Hom}_\Lambda(\Lambda e, M) \cong eM$$

becomes an isomorphism of left $e\Lambda e$ -modules.

- (c) [2, Exercise 12.4 (b)] Suppose $\Lambda = k\Gamma/I$ for some quiver Γ and admissible ideal I , and let $i \in \Gamma_0$ be a vertex. Show that $\dim_k \mathrm{Hom}_\Lambda(\Lambda e_i, M) = \dim_k M(i)$, where $M(i)$ is the vector space at vertex i when M is considered as a representation.

3. [1, Exercise I.7.7] let $\Lambda = k[t]$. Show that the Λ -module $k[t]/(t^3)$ has no projective cover in $\text{Mod } \Lambda$
4. [2, Exercise 12.3] Let Λ be a ring and M a module. A *projective resolution* of M is a sequence

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$$

where each P_i is projective, $\ker(f_i) = \text{Im}(f_{i+1})$ for all $i \geq 0$, and f_0 is surjective. Often we use the symbol P_\bullet to denote a projective resolution.

If there exists some m so that $P_m \neq 0$ and $P_n = 0$ for all $n > m$, then we say that the length of the projective resolution is m . The *projective dimension* of M is then defined to be the minimum possible length of a projective resolution for M .

There is a theorem which says that if P_\bullet is a projective resolution of M so that $P_0 \xrightarrow{f_0} M$ is a projective cover and $P_{i+1} \xrightarrow{f_{i+1}} \ker(f_i)$ is a projective cover for all $i \geq 0$, then the projective dimension of M is equal to the length of P_\bullet .

- (a) Let $\Gamma = 1 \longrightarrow 2$ and let $\Lambda = k\Gamma$. Find the projective dimensions of the modules $\Lambda e_1, \Lambda e_2$, and $S_1 = \Lambda e_1 / \Lambda e_2$.
- (b) Let $\Gamma = 1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2$ and let $\Lambda = k\Gamma / (\alpha\beta, \beta\alpha)$. Find the projective dimensions of the simple modules S_1 and S_2 .
- (c) For an arbitrary ring Λ and module M , show that the projective dimension of M is 0 if and only if M is projective.
- (d) Let Λ be a left artinian ring. The *left global dimension* of Λ is the supremum of the projective dimensions of all finitely generated left Λ -modules¹. Show that if Λ is *left hereditary* (that is, every submodule of a finitely generated projective left module is projective) then the global dimension of Λ is at most 1.²
- (e) (Challenge) Let Γ be a finite acyclic quiver. For an arrow $\alpha \in \Gamma_1$ let $s(\alpha)$ and $t(\alpha)$ denote its source and target of α , respectively.

¹It can be shown that this is the same as the supremum of all Λ -modules.

²The converse to this statement is also true.

Show that a finite-dimensional representation (V, f) of Γ is projective if and only if the map

$$\bigoplus_{\alpha \in \Gamma_1, t(\alpha)=i} V_{s(\alpha)} \xrightarrow{(f_\alpha)} V_i$$

is a monomorphism for all $i \in \Gamma_0$. Deduce that $k\Gamma$ is left hereditary.

References

- [1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).
- [2] E. Hanson, 2021 MA3203 Problem Sheets, NTNU, https://wiki.math.ntnu.no/ma3203/2021v/course_schedule.