MA3203 - Problem Set 12 Hints and Answers

(3a) The projective dimensions of Λe_1 and Λe_2 are 0. The projective dimension of S_1 is 1.

(3b) Both simple modules have projective dimension infinity.

(3d) Suppose that Λ is left hereditary and let M be an arbitrary left Λ-module. Then there exists a free module F an a surjection $F \xrightarrow{f} M$. Now since Λ is left hereditary, we know that $\ker(f)$ is projective and so $0 \to \ker(f) \to F \to M \to 0$ is a projective resolution for M of length 1.[1] We conclude that the projective dimension of M is at most 1.

(3e) Suppose that Λ has left global dimension 0. This means every left Λ-module is projective. Now since Λ is finitely generated, it contains a maximal submodule $M \subseteq \Lambda$. We then have that $\Lambda/M \cong S$ is simple and projective. This means $\Lambda \cong S \oplus M$, so in particular Λ contains a simple submodule.

Now let N be the sum (and hence the direct sum) of all of the simple submodule of Λ. Then $\Lambda \cong N \oplus (\Lambda/N)$ since Λ/N is projective. Moreover, we have tthat Λ/N contains no simple submodule by assumption. However, any nonzero finitely generated submodule of Λ/N will contain a simple submodule by the same argument as before. This means $\Lambda/N = 0$ and so Λ is semisimple.

Now suppose that Λ is semisimple and let M be a left Λ-module. Then M is isomorphic to a direct sum of simple modules, all of which are projective. In particular, M has projective dimension 0.

(4a) *Hint: Suppose* $\mathrm{Hom}(P, M) \neq 0$. *Since M has finite length, there exists a nonnegative integer m so that* $\mathrm{Hom}_\Lambda(P, \mathrm{rad}^m M) \neq 0$ *and* $\mathrm{Hom}_\Lambda(P, \mathrm{rad}^{m+1} M) = 0$. *This means that $P/\mathrm{rad}P$ is a direct summand of* $\mathrm{rad}^m M/\mathrm{rad}^{m+1} M$. *In particular, there exists* $\mathrm{rad}^{m+1} M \subseteq N \subseteq \mathrm{rad}^m M$ *so that* $N/\mathrm{rad}^{m+1} M \cong P/\mathrm{rad}P$. *Now argue that N and* rad^{m+1} *appear together in a composition series for M.*

For the reverse direction, use the fact that P is projective to show that if $N/L \cong P/\mathrm{rad}P$ *then* $\mathrm{Hom}_\Lambda(P, N) \neq 0$.

(4b) *Hint: Recall that $M(i) = e_i M$. Now consider* $\phi : \mathrm{Hom}_\Lambda(\Lambda e_i, M) \to e_i M$ *given by* $\phi(f) = f(e_i)$. *Since e_i is idempotent, we have* $\phi(f) = f(e_i) = f(e_i^2) = e_i f(e_i) \in e_i M$, *so ϕ is well defined as a map on sets. Show that ϕ is in fact an isomorphism of vector spaces.*

[1] When a projective resolution has finite length, it is usually not written as an infinite series of morphisms. So in this particular example, the projective resolution we are referring to has $P_i = 0$ for all $i \geq 2$.