MA3203 - Problem Set 12 (Projectives)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise stated.

1. Let $Q = \;1 \rightrightarrows 2\;$. For $\lambda \in K$ define a representation $M(\lambda) = \;K \underset{\lambda}{\overset{1}{\rightrightarrows}} K\;$ and define $M(\infty) = \;K \underset{1}{\overset{0}{\rightrightarrows}} K\;$. For all $\lambda \in K \cup \{\infty\}$, show that there is a projective cover $\Lambda e_1 \xrightarrow{f_\lambda} M(\lambda)$ and find $\ker(f_\lambda)$.

2. Let Λ be a ring. A short exact sequence of Λ-modules

$$0 \longrightarrow M \xrightarrow{\;f\;} N \xrightarrow{\;g\;} L \longrightarrow 0$$

is called *split* if the three equivalent conditions hold:

- There exists $g' : L \to N$ so that $g \circ g' = 1_L$. In this case we say g admits a *section.*
- There exists $f' : N \to M$ so that $f' \circ f = 1_M$. In this case, we say f admits a *retraction.*
- There is an isomorphism $h : N \to M \oplus L$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\;f\;} & N & \xrightarrow{\;g\;} & L & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 1_M} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1_L} & & \downarrow \\
0 & \longrightarrow & M & \xrightarrow{\begin{bmatrix} 1_M & 0 \end{bmatrix}} & M \oplus L & \xrightarrow{\begin{bmatrix} 0 \\ 1_L \end{bmatrix}} & L & \longrightarrow & 0
\end{array}$$

Show that a Λ-module P is projective if and only if every short exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\;f\;} N \xrightarrow{\;g\;} P \longrightarrow 0$$

is split.

3. Let Λ be a ring and M a module. A *projective resolution* of M is a sequence

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

where each P_i is projective, $\ker(f_i) = \mathrm{Im}(f_{i+1})$ for all $i \geq 0$, and f_0 is surjective. Often we use the symbol $P_\bullet$ to denote a projective resolution.

If there exists some m so that $P_m \neq 0$ and $P_n = 0$ for all $n > m$, then we say that the length of the projective resolution is m. The *projective dimension* of M is then defined to be the minimum possible length of a projective resolution for M.

There is a theorem which says that if $P_\bullet$ is a projective resolution of M so that $P_0 \xrightarrow{f_0} M$ is a projective cover and $P_{i+1} \xrightarrow{f_{i+1}} \ker(f_i)$ is a projective cover for all $i \geq 0$, then the projective dimension of M is equal to the length of $P_\bullet$.

(a) Let $Q = \; 1 \longrightarrow 2 \;$ and let $\Lambda = KQ$. Find the projective dimensions of the modules $\Lambda e_1, \Lambda e_2$, and $S_1 = \Lambda e_1/\Lambda e_2$.

(b) Let $Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \;$ and let $\Lambda = KQ/(\alpha\beta, \beta\alpha)$. Find the projective dimensions of the simple modules S_1 and S_2.

(c) For an arbitrary ring Λ and module M, show that the projective dimension of M is 0 if and only if M is projective.

(d) The *left global dimension* of a ring Λ is the supremum of the projective dimensions of all left Λ-modules[1]. Show that if Λ is *left hereditary* (that is, every submodule of a projective left module is projective) then the global dimension of Λ is at most 1.[2]

(e) (Challenge) Show that the left global dimension of a ring Λ is equal to 0 if and only if Λ is semisimple. *Hint: If every module is projective, then every submodule is a direct summand.*

4. Let Λ be a finite dimensional K-algebra and let M be a Λ-module of finite length.

(a) [1, Exercise II.1a] Let P be an indecomposable projective module. Show that $\mathrm{Hom}_\Lambda(P, M) \neq 0$ if and only if $P/\mathrm{rad}P$ is a composition factor of M.

(b) (Challenge) Suppose $\Lambda = KQ/I$ for some quiver Q and admissible ideal I, and let $i \in Q_0$ be a vertex. Recall that $\mathrm{Hom}_\Lambda(\Lambda e_i, M)$ is a vector space over K. Show that $\dim_K \mathrm{Hom}_\Lambda(\Lambda e_i, M) = \dim_K M(i)$, where $M(i)$ is the vector space at vertex i when M is considered as a representation.

References

[1] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press (1995).

[1] If Λ is Noetherian, then this is the left and right global dimension of Λ are the same and this value is simply referred to as the global dimension.

[2] The converse to this statement is also true, and we will prove this later in the semester.