MA3203 - Problem Set 11 Hints and Answers

(2) We will first show that a surjective morphism of R-modules $f : M \to N$ is an essential epimorphism if and only if $\ker(f) \subseteq M$ is small (even without assuming that M and N are finitely generated).

Suppose $\ker(f)$ is small and consider $g : L \to M$ so that $f \circ g$ is surjective. This means

$$N \cong \operatorname{Im}(f \circ g) \cong \operatorname{Im}(g)/(\ker(f) \cap \operatorname{Im}(g)) \cong (\operatorname{Im}(g) + \ker(f))/\ker(f).$$

We conclude that $\operatorname{Im}(g) + \ker(f) = M$, and so $\operatorname{Im}(g) = M$.

Now suppose $\ker(f)$ is not small and write $M = \ker(f) + L$ with $L \subsetneq M$. As before, we can conclude that $N \cong L/(L \cap \ker f)$. Thus if $\iota : L \hookrightarrow M$ is the inclusion map, we have that $f \circ \iota$ is surjective and ι is not.

Returning to our context, we have that $P \oplus Q$ is projective if and only if both P and Q are projective since projective modules are direct summands of free modules. Now let $K_f = \ker f$ and $K_g = \ker g$. Then it remains to show that $K_f \oplus K_g \subseteq P \oplus Q$ is small if and only if $K_f \subseteq P$ and $K_g \subseteq Q$ are both small. We do this in two steps

1. $K_f \oplus K_g \subseteq P \oplus Q$ is small if and only if both $K_f \oplus 0 \subseteq P \oplus Q$ and $0 \oplus K_g \subseteq P \oplus Q$ are small.
2. $K_f \oplus 0 \subseteq P \oplus Q$ is small if and only if $K_f \subseteq P$ is small (and likewise for K_g and Q).

Proof of Step 1: First suppose that $K_f \oplus 0$ and $0 \oplus K_g$ are small and consider $L \subseteq M$ so that $K_f \oplus K_g + L = P \oplus Q$. Then $(K_f \oplus 0) + ((0 \oplus K_g) + L) = M$ and so $(0 \oplus K_g) + L = P \oplus Q$ since $K_f \oplus 0$ is small. This then implies that $L = P \oplus Q$ since $0 \oplus K_g$ is small. Now suppose that $K_f \oplus K_g$ is small and consider $L \subseteq P \oplus Q$ so that $(K_f \oplus 0) + L = P \oplus Q$. Then $(K_f \oplus K_g) + L = P \oplus Q$, and so $L = P \oplus Q$ since $K_f \oplus K_g$ is small. The argument that $0 \oplus K_g$ is small is identical.

Proof of Step 2: First suppose that $K_f \subseteq P$ is small and consider $L \subseteq P \oplus Q$ so that $(K_f \oplus 0) + L = P \oplus Q$. Then $(K_f \oplus 0) + L \cap (P \oplus 0) = P \oplus 0$, and that fact that K_f is small in P then implies that $L \cap (P \oplus 0) = P \oplus 0$. This means $(K_f \oplus 0) \subseteq L$ and so $L = P \oplus Q$. Now suppose that $(K_f \oplus 0) \subseteq P \oplus Q$ is small and consider $L \subseteq P$ so that $K_f + L = P$. Then $(K_f \oplus 0) + (L \oplus Q) = P \oplus Q$, and so $L \oplus Q = P \oplus Q$ since $(K_f \oplus 0)$ is small. This means $L = P$.

(3a) The projective cover is $\Lambda e_1 \oplus \Lambda e_2 \oplus \Lambda e_3 = \Lambda$. More precisely, we should also specify the morphism, so the projective cover is really

$$\begin{array}{ccccccc}
\Lambda : & & K & \xrightarrow{[1\ 0]} & K^2 & \xrightarrow{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}} & K^3 \\
\Big\downarrow p & & \Big\downarrow 1 & & \Big\downarrow [0\ 1] & & \Big\downarrow [0\ 0\ 1] \\
M : & & K & \xrightarrow{0} & K & \xrightarrow{0} & K.
\end{array}$$

It can in particular be shown that the induced morphism on the tops of Λ and M is an isomorphism.

The kernel of the projective cover is

$$0 \longrightarrow K \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} K^2,$$

which is isomorphic to $\Lambda e_2 \oplus \Lambda e_3$.

(4b) Note: this representation is indecomposable, so the result of Problem (2) cannot be used to simplify the calculation.

The projective cover is $\Lambda e_1 \oplus \Lambda e_2$. More precisely, we should also specify the morphism, so the projective cover is really

$$\begin{array}{ccccccc}
\Lambda e_1 \oplus \Lambda e_2 : & & K & \xrightarrow{[0\ 1]} & K^2 & \overset{\begin{bmatrix}1&0\\0&0\\0&1\\0&0\end{bmatrix}}{\underset{\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}}{\rightrightarrows}} & K^4 \\
\Big\downarrow p & & \Big\downarrow 1 & & \Big\downarrow \begin{bmatrix}1&0\\0&1\end{bmatrix} & & \Big\downarrow [1\ 1\ 0\ 1] \\
M : & & K & \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} & K^2 & \overset{[1\ 0]}{\underset{[1\ 1]}{\rightrightarrows}} & K
\end{array}$$

The tops of both $\Lambda e_1 \oplus \Lambda e_2$ and M in this case are isomorphic to $S_1 \oplus S_2$. The kernel of the projective cover is $0 \longrightarrow 0 \longrightarrow K^3$, which is isomorphic to $(\Lambda e_3)^3$.

(4c) Let M be the representation in question. Then M is isomorphic to the direct sum of 3 indecomposable representations:

$$M \cong \ (0 \longrightarrow K \overset{1}{\underset{1}{\rightrightarrows}} K) \oplus (0 \longrightarrow K \rightrightarrows 0) \oplus (0 \longrightarrow 0 \rightrightarrows K).$$

By Problem 2, the projective cover of M is the direct sum of the projective covers of each of these indecomposable pieces. It can then be shown that the projective cover is of the form $((\Lambda e_2)^2 \oplus \Lambda e_3) \xrightarrow{f} M$ and that $\ker(f) \cong (\Lambda e_3)^3$.