MA3203 - Problem Set 9 Answers and Hints

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise specified.

(1a) $\text{rad}(\Lambda e_1) \cong \Lambda e_2$

(1b) $\text{rad}(\Lambda e_2) \cong S_1 \oplus S_3$.

(1c) $\text{rad}(\Lambda(e_1 + e_3)) \cong S_2 \oplus S_2$.

(1d) $\text{rad}(\mathbb{Z}/p^2\mathbb{Z})$ is the submodule generated by $p + p^2\mathbb{Z}$, which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(2a) Note that $\text{rad}M \cong M$ if and only if $M = 0$. Thus, since $\ell(M) < \infty$, there exists some minimum $m \in \mathbb{N}$ so that $\text{rad}^m M = 0$. This gives a strictly increasing series of submodules

$$rad^m M = 0 \subsetneq \text{rad}^{m-1} M \subsetneq \cdots \subsetneq \text{rad}M \subsetneq M$$

We conclude that $m \leq \ell(M)$.

(2b) Suppose $m = \ell(M)$. Then the series of submodules described in (2a) must be a composition series for M. In particular, we have that $M/\text{rad}M$ is simple, and so $\text{rad}M$ is a maximal submodule (and therefore the unique maximal submodule) of M. Repeating this argument, we see that $\text{rad}^i M$ is the unique maximal submodule of $\text{rad}^{i-1} M$ for all $i \leq m$. In particular, our composition series contains all of the submodules of M, and is therefore unique.

Now suppose M is uniserial and let

$$M = M_0 \subsetneq M_1 \subseteq \cdots \subsetneq M_{\ell(M)} = M$$

be its unique composition series. It follows that M_i is the unique maximal submodule of M_{i+1} for all $i < \ell(M)$. In particular, this means $M_i = \text{rad}M_{i+1}$ for all $i < \ell(M)$. The result then follows immediately.