MA3203 - Problem Set 9 (Radicals)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise specified.

1. (a) Let $Q = \ 1 \longrightarrow 2 \longrightarrow 3$ and let $\Lambda = KQ$. Find the radical of the representation corresponding to the module Λe_1.
   (b) Let $Q = \ 1 \longleftarrow 2 \longrightarrow 3$ and let $\Lambda = KQ$. Find the radical of the representation corresponding to the module Λe_2.
   (c) Let $Q = \ 1 \longrightarrow 2 \longleftarrow 3$ and let $\Lambda = KQ$. Find the radical of the representation corresponding to the module $\Lambda(e_1 + e_3)$.
   (d) Let p be a prime. Considering $\mathbb{Z}/p^2\mathbb{Z}$ as a $\mathbb{Z}$-module), what is $\mathrm{rad}(\mathbb{Z}/p^2\mathbb{Z})$?

2. Let Λ be a ring and let M be a module of finite length.
   (a) Show that $\mathrm{rad}^{\ell(M)} M = 0$.
   (b) Recall that M is called uniserial if it has a unique composition series. Let m be the smallest positive integer so that $\mathrm{rad}^m M = 0$ (this is sometimes called the *radical length* of M). Show that M is uniserial if and only if its radical length is equal to its length.
   (c) Find the radical length of the modules in exercise 1.

3. Let Λ be a ring and let M be a module of finite length. Recall that a submodule $N \subseteq M$ is called small (also known as superfluous) if there does not exist a proper submodule $X \subsetneq M$ so that $M = X + N$.
   (a) Let $N \subseteq M$ be small and let $L \subseteq M$ be maximal. Show that $N \subseteq L$.
   (b) Conclude that
   $$\mathrm{rad} M = \sum_{N \subseteq M:\ N \text{ is small}} N.$$
   Hint: we already know that $\mathrm{rad} M$ *is small in* M.

4. Let Λ be a ring and let M, N be modules. Show that $\mathrm{rad}(M \oplus N) = \mathrm{rad}(M) \oplus \mathrm{rad}(N)$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).