MA3203 - Problem Set 10 Answers and Hints

Note: The videos on the Krull-Remak-Schmidt Theorem include a detailed proof of the result in Problem 3.

(2a) The radical is $S_2 \oplus S_3$, the top is S_1, and the annihilator is $\Lambda\beta$.

(3b) Suppose e is invertible. Then $e = e^{-1}ee = e^{-1}e = 1$. The proof for $(1-e)$ is similar.

(3e) Let $x \in \ker(f^m) \cap \operatorname{Im}(f^m)$ and suppose $x = f^m(y)$. Then $0 = f^m(x) = f^{2m}(y)$ implies that $y \in \ker(f^{2m}) = \ker(f^m)$. We conclude that $x = 0$.

(3f) Let $x \in M$. Since $\operatorname{Im}(f^m) = \operatorname{Im}(f^{2m})$, there exists $y \in M$ such that $f^m(x) = f^{2m}(y)$. This means $x - f^m(y) \in \ker(f^m)$. As $f^m(y) \in \operatorname{Im}(f^m)$, we thus have $x \in \ker(f^m) + \operatorname{Im}(f^m)$.

(3h) Let m be the smallest integer so that $f^m = 0$ and suppose f^{-1} is the inverse of f. Then $f^{m-1} = f^{-1}f^m = 0$, a contradiction. The proof for $g \circ f$ is similar.

(3i) The inverse of $Id_M - g \circ f$ is $Id_M + \sum_{i=1}^{m'-1}(g \circ f)^i$.