MA3203 - Problem Set 10 (Radicals)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise specified.

1. Let Λ be a ring and let $f : M \to N$ and $g : N \to L$ be essential epimorphisms of Λ-modules. Show that $g \circ f : M \to L$ is an essential epimorphism.

2. [1, Exercise III.9ab] Find the radical, top, and annihilator of each of the following representations of the quiver $\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 \end{tikzcd}$.

   (a) $$\begin{tikzcd} K \arrow[r, "1"] \arrow[rr, bend left, "1"] & K \arrow[r, "0"] & K \end{tikzcd}$$

   (b) $$\begin{tikzcd} K \arrow[r, "0"] \arrow[rr, bend left, "1"] & K \arrow[r, "1"] & K \end{tikzcd}$$

   (c) $$\begin{tikzcd} K \arrow[r, "1"] \arrow[rr, bend left, "1"] & K \arrow[r, "1"] & K \end{tikzcd}$$

3. Let Λ be a ring and let M be a Λ-module of finite length. Recall that the *endomorphism ring* of M is $\mathrm{End}(M) := \mathrm{Hom}_\Lambda(M, M)$. Addition in $\mathrm{End}(M)$ is defined pointwise (so $(f + g)(x) := f(x) + g(x)$) and multiplication is given by function composition.

   The goal of this exercise is to show that M is indecomposable if and only if $\mathrm{End}(M)$ is a local ring.

   (a) Recall that a local ring is one with a unique maximal ideal. Equivalently, it is a ring for which all non-invertible elements form an ideal. Show that an arbitrary ring R is local if and only if $\mathrm{rad}R$ is a maximal ideal.

   (b) Suppose that M is not indecomposable. Thus by a previous problem set, there exists an element $e \in \mathrm{End}(M)$ so that $e^2 = e$ and $0 \neq e \neq 1$. Show that neither e nor $1 - e$ is invertible.

   (c) Conclude that if M is not in indecomposable, then $\mathrm{End}(M)$ is not local.

   (d) Now suppose that M is indecomposable and let $f \in \mathrm{End}(M)$. Show that there exists some positive integer m so that $\ker(f^m) = \ker(f^t)$ and $\mathrm{Im}(f^m) = \mathrm{Im}(f^t)$ for all $t \geq m$. *Hint: remember that M has finite length!*

(e) Show that $\ker(f^m) \cap \mathrm{Im}(f^m) = 0$. *Hint: suppose x is in the intersection and write $x = f^m(y)$. What can be said about $f^{2m}(y)$?*

(f) (Challenge) Show that $\ker(f^m) + \mathrm{Im}(f^m) = M$.

(g) Conclude that $M \cong \mathrm{Im}(f^m) \oplus \ker(f^m)$. Thus since M is indecomposable, either $f^m = 0$ (so f is nilpotent) or f is an isomorphism[1].

(h) It remains to show that if f is nilpotent, then $f \in \mathrm{radEnd}(M)$. Thus suppose f is nilpotent and let $g \in \mathrm{End}(M)$ be arbitrary. Show that neither f nor $g \circ f$ is (left) invertible.

(i) Conclude that $g \circ f$ is nilpotent, so that there exists some m' with $(g \circ f)^{m'} = 0$. Use this to show that $Id_M - g \circ f$ is invertible (where Id_M is the identity map on M).

(j) Conclude that if M is indecomposable then $\mathrm{End}(M)$ is local, and congratulate yourself on making it through this problem!

References

[1] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press (1995).

[1] This result is known as Fitting's Lemma.