MA3203 - Problem Set 8 (Radicals)

In all problems, K denotes a field, all representations are assumed to be finite dimensional representations over K, and all ideals are two-sided unless otherwise specified.

1. Show that $\mathrm{rad}K[x_1, \dots, x_n] = 0$ for all $n \geq 1$.

2. Let Q be an acyclic quiver and let $\Lambda = KQ$. Recall that for an arbitrary $x \in \Lambda$, we have $x \in \mathrm{rad}\Lambda$ if and only if $1 - yx$ is left invertible for all $y \in \Lambda$.

   (a) Show that $e_i \notin \mathrm{rad}\Lambda$ for any $i \in Q_0$. *Hint: For any $y \in \Lambda$, we know $y(1 - e_i)$ is a linear combination of paths in Q. What can be said about these paths?*

   (b) Let $\alpha \in Q_1$. Show that $\alpha \in \mathrm{rad}\Lambda$. *Hint: For any $y \in \Lambda$, what can we say about $(y\alpha)^2$?*

   (c) Conclude that $\mathrm{rad}\Lambda = (Q_1)$, the ideal generated by the set of arrows.

3. Find an example of a quiver Q for which $\mathrm{rad}KQ \neq (Q_1)$.

4. Let (Q, ρ) be a quiver with relations and let $\Lambda = KQ/(\rho)$. Suppose that ideal (ρ) is admissible; i.e., that there exists $m > 0$ so that $(Q_1)^m \subseteq (\rho) \subseteq (Q_1)^2$.

   (a) Show that $e_i \notin \mathrm{rad}\Lambda$ for any $i \in Q_0$.

   (b) Let $\alpha \in Q_1$. Show that $\alpha \in \mathrm{rad}\Lambda$. *Hint: can the fact that (ρ) is admissible be used to generalize your argument in (3b)?*

   (c) Conclude that $\mathrm{rad}\Lambda = (Q_1)$, or more precisely that $\mathrm{rad}\Lambda = (Q_1)/(\rho)$.

5. [1, Exercise I.1] Let $f : \Lambda \to \Lambda'$ be a surjective morphism of rings. Show that $f(\mathrm{rad}\Lambda) \subseteq \mathrm{rad}\Lambda'$.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts 65, Cambridge Univ. Press (2006).